Effect of sloppy discretization on reflection and transmission coefficients

Jukka Tuomela

To cite this version:

Jukka Tuomela. Effect of sloppy discretization on reflection and transmission coefficients. [Research Report] RR-1537, INRIA. 1991. inria-00075025

HAL Id: inria-00075025
https://inria.hal.science/inria-00075025

Submitted on 24 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Rapports de Recherche

N° 1537

Programme 6
Calcul Scientifique, Modélisation et
Logiciel numérique par Ordinateur

EFFECT OF SLOPPY DISCRETIZATION ON REFLECTION AND TRANSMISSION COEFFICIENTS

Jukka TUOMELA

Octobre 1991

RR_1537

Effect of Sloppy Discretization on Reflection and Transmission Coefficients

l'Effet de Discrétisation Paresseuse sur les Coefficients de Réflexion et Transmission

Jukka Tuomela

INRIA, Rocquencourt
BP 105
78153 Le Chesnay Cedex
France

Helsinki University of Technology
02150 Espoo
Finland

Abstract

When hyperbolic problems are discretized it is natural to use regular grids. The boundaries usually do not respect this regularity; however, it is tempting to deform the boundary in such a way that the regular gird can be used everywhere. Introducing a simple model problem we study the effect of this deformation on the reflection and transmission coefficients.

Résumé

Dans la résolution des problèmes hyperboliques par différences ou éléments finis, il est naturel d'utiliser un maillage régulier. Or, cette régularité est difficile à obtenir si les frontières ne sont pas régulières. Pour avoir des schémas simples on pourrait cependant déformer la frontière de telle façon que le maillage reste régulier partout. Nous introduisons un problème simple pour étudier l'effet de cette déformation sur les coefficients de réflexion et transmission.

1 Introduction

When discretizing hyperbolic equations one usually uses finite difference methods or finite elements with regular mesh to minimize the effect of dispersion. However, it is usually impossible to fit this regular grid into the domain under consideration, so there are two possibilities: either to choose to be sloppy and deform the domain such that regular mesh can be used everywhere or use irregular mesh near the boundary. The latter case leads to isoparametric elements if finite elements are used or to some ad hoc adjustments in case of finite differences. We will not treat this situation. Instead we will take up the former possibility and try to analyse what is the effect of the sloppy discretization on the numerical solution.

These kind of questions arise notably in scattering problems, see [WI]. Take a simply connected compact domain K (obstacle) and solve the Helmholtz (or wave or Maxwell) equation in $\mathbb{R}^n \setminus K$ with appropriate boundary conditions ($n = 2$ or 3 in applications). Of course it would be difficult to do explicit calculations with this model so we choose a simple model problem where analytical solutions are available. The simplest case is evidently the reflection of the plane wave from the straight line. Then we 'simulate' the sloppy discretization by not aligning the mesh and the line; this gives a kind of staircase boundary. In the same way we can treat the transmission problem (in fact the pure reflection problem is a limiting case of the more general reflection transmission problem).

We calculate the reflection and transmission ceofficients for Q_1 elements, P_1 elements (with equilateral triangles) and for the five points scheme. In all cases it is found that the staircase introduces errors of first order although all the schemes are of second order. This seems to indicate that the sloppy discretization should be avoided in practical calculations.

2 Review of Continuous Case

Consider the following problem: solve the wave equation

$$u_{tt} - \Delta u = 0 \tag{2.1}$$

in the domain $\{\,(x_1, x_2)\,|\,x_2 > f(x_1)\,\}$ where f is some periodic function with period δ. At the boundary we put $u = 0$ (for instance). Then consider the incident wave $u_i = \exp(i(k_1x_1 - k_2x_2 - \omega t))$ where $|k| = \omega$. We want to determine the scattered wave u_s. To this end we first note that the periodicity of f imposes a 'pseudoperiodicity' of u_s, that is

$$u_s(x_1 + \delta, x_2, t) = e^{ik_1\delta}u_s(x_1, x_2, t). \tag{2.2}$$

Guided by this observation we can then look for the u_s in the following form

$$u_s(x_1, x_2, t) = \sum_{n=-\infty}^{\infty} R_n e^{i(a_n x_1 + b_n x_2 - \omega t)}$$

where $a_n = k_1 + 2\pi n/\delta$ and $b_n = \sqrt{|k|^2 - a_n^2}$. Evidently when $|n| \to \infty$ sooner or later $|k|^2 - a_n^2$ becomes negative; in that case we define $b_n = i\sqrt{a_n^2 - |k|^2}$. The above sum is sometimes called Rayleigh series and the terms Floquet modes (for a 'physical' treatment of the scattering by periodic structures see for instance [KO]). So all in all we have a finite number of propagating modes and an infinite number of evanescent modes. Then taking α be the angle between the incident wave and x_2-axis, we have the following simple result

Proposition 2.1 *The n'th mode is propagating if*

$$-\frac{(1 + \sin\alpha)\delta}{\lambda} \le n \le \frac{(1 - \sin\alpha)\delta}{\lambda}$$

where λ is the wavelength.

Proof Putting $2\pi/\delta = p$ the condition is

$$(k_1 + pn)^2 \le |k|^2 = k_1^2 + k_2^2.$$

Solving this and recalling that $k_1 = |k| \sin\alpha$ and $\lambda = 2\pi/|k|$ gives the result.

■

The coefficients R_n are determined by using the same kind of expansion for $u_i(x_1, f(x_1))$ and matching the corresponding coefficients. However, let us note that in general the expansion does not converge in the grooves, that is when $x_2 < \max f$. Anyway, if one is interested only in the solution far away from the boundary, the analysis based on the expansion is valid: we can ignore the evanescent modes and take into account only the propagating modes. As we have seen there are only a finite number of them so there are no problems of convergence.

Note that the time dependence was not really essential, so that we could have made our analysis directly in terms of the Helmholtz equation

$$\omega^2 u + \Delta u = 0.$$

This remains true in the discrete case, and consequently from now on we consider only the Helmholtz equation.

The transmission problem can be treated similarly: we just have different expansions 'above' and 'below' and then match the coefficients at the interface. Adopting the notation $\Omega_1 = \{(x_1, x_2) \,|\, x_2 > f(x_1)\}$, $\Omega_2 = \{(x_1, x_2) \,|\, x_2 < f(x_1)\}$ and

$\Gamma = \{ (x_1, x_2) \,|\, x_2 = f(x_1) \}$, the Helmholtz equation can be written as

$$\begin{cases} \rho_1 \omega^2 u + \nabla \cdot (\mu_1 \nabla u) = 0 & \text{in } \Omega_1 \\ \rho_2 \omega^2 u + \nabla \cdot (\mu_2 \nabla u) = 0 & \text{in } \Omega_2. \end{cases} \tag{2.3}$$

Instead of ρ_i and μ_i it is often more convenient to use the parameters $c_i = \sqrt{\mu_i/\rho_i}$ (wave speed) and $z_i = \sqrt{\mu_i \rho_i}$ ((characteristic) impedance). We will suppose that ρ_i and μ_i are constant, but obviously this need not be the case in general.
Then recall that taking Ω_1 to be the upper half plane there is only one reflected and one transmitted wave, so that we look for solutions of the form

$$u(x) = \begin{cases} \exp(k_1 x_1 - k_2 x_2) + R \exp(k_1 x_1 + k_2 x_2) & x_2 \geq 0 \\ T \exp(\xi_1 x_1 + \xi_2 x_2) & x_2 \leq 0 \end{cases}$$

where ξ is the wave vector of the transmitted wave. The continuity considerations at the interface give (see [KO])

$$\begin{aligned} \xi_1 &= k_1 \\ c_1 \, |k| &= c_2 \, |\xi| \end{aligned}$$

and consequently

$$\xi_2 = -\frac{|k|}{c_2} \sqrt{c_1^2 - c_2^2 \sin^2 \alpha}.$$

The minus sign appears because the transmitted signal goes 'down'. We see that when $c_1 < c_2 \sin \alpha$ there is no propagating mode in the lower half plane.
Then elementary calculations show that the reflection and transmission coefficients are given by

$$\begin{aligned} R &= \frac{c_1 z_1 \cos \alpha - z_2 \sqrt{c_1^2 - c_2^2 \sin^2 \alpha}}{c_1 z_1 \cos \alpha + z_2 \sqrt{c_1^2 - c_2^2 \sin^2 \alpha}} \\ T &= \frac{2 c_1 z_1 \cos \alpha}{c_1 z_1 \cos \alpha + z_2 \sqrt{c_1^2 - c_2^2 \sin^2 \alpha}} \end{aligned} \tag{2.4}$$

where α is the incident angle. Note that putting $z_2 = 0$ one gets the Neumann boundary condition and letting z_2 tend to infinity yields the Dirichlet condition.

3 Reflection

We consider the following model problems (see figures 3.1 and 3.2): in the first we discretize the problem with Q_1 elements and in the second with P_1 elements using

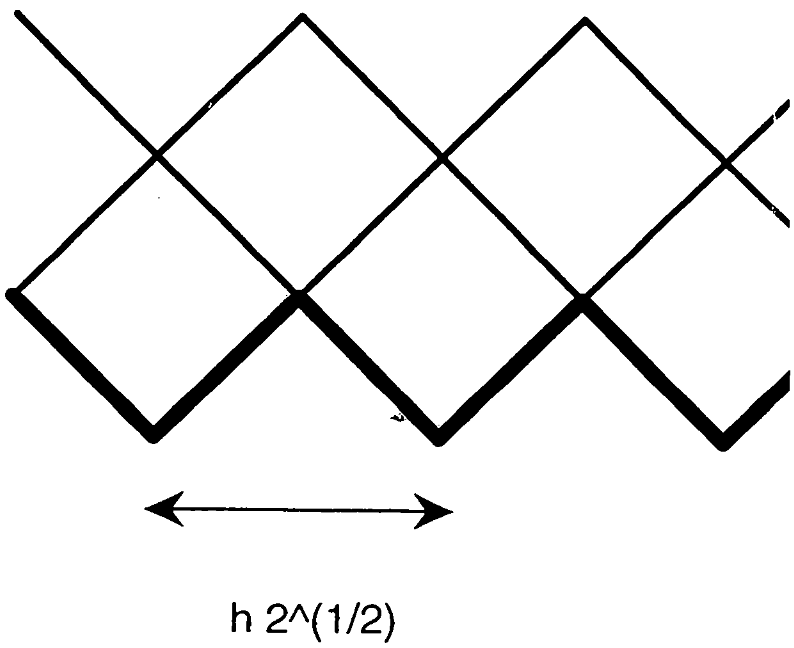

Figure 3.1: Q_1 grid.

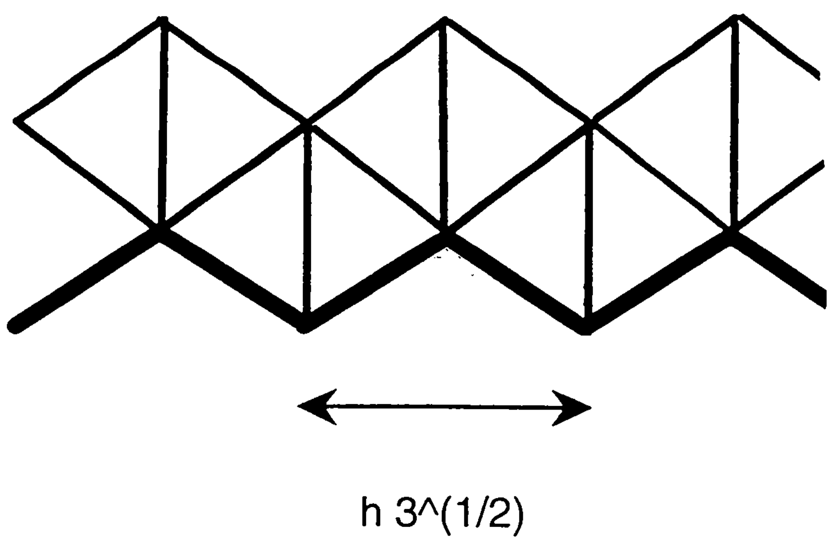

Figure 3.2: P_1 grid.

equilateral triangles. In the first case the period is then $\delta = \sqrt{2}h$ and in the second $\delta = \sqrt{3}h$.

In both cases there are only two (essentially inequivalent) boundary conditions so we expect that two reflecting modes are sufficient. Let us introduce the following notations: denote the incident wave vector by $\bar{k} = \begin{pmatrix} k_1 & -k_2 \end{pmatrix}$ (where it is implicitly assumed that k_1 and k_2 are positive); then $k = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$ stands for the wave vector of the principal reflected mode and $\tilde{k} = \begin{pmatrix} \tilde{k}_1 & \tilde{k}_2 \end{pmatrix}$ is the wave vector of the parasitic mode. Note that $\tilde{k}_2$ might be a complex number in which case its imaginary part must be positive. The reflection coefficient of the principal mode will be denoted by R and that of the parasitic mode by $\tilde{R}$, so that we look for the solution of the form

$$u(x) = \exp(i\bar{k}\cdot x) + R\exp(ik\cdot x) + \tilde{R}\exp(i\tilde{k}\cdot x). \tag{3.1}$$

After these preliminaries we then take up the Q_1 case.

3.1 Q_1 Element

Then recalling the familiar Q_1 scheme and 'rotating it by 45 degrees' we get the following difference equation.

$$\begin{aligned} 8u_{m,n} - u_{m+1,n} - u_{m-1,n} - u_{m,n+1} - u_{m,n-1} - u_{m+1/2,n+1/2} & \\ -u_{m+1/2,n-1/2} - u_{m-1/2,n+1/2} - u_{m-1/2,n-1/2} &= 3\omega^2 h^2 u_{m,n} \end{aligned}$$

where $u_{m,n} = u(mh\sqrt{2}, nh\sqrt{2})$ and $m+n$ is an integer and m and n are either integers or integers divided by two. Then substituting the principal mode $\exp(ik\cdot x)$ and the parasitic mode $\exp(i\tilde{k}\cdot x)$ we get the following dispersion relations

$$\begin{aligned} 8 - 2\cos(\sqrt{2}k_1 h) - 2\cos(\sqrt{2}k_2 h) - 4\cos(k_1 h/\sqrt{2})\cos(k_2 h/\sqrt{2}) &= 3\omega^2 h^2 \\ 8 - 2\cos(\sqrt{2}\tilde{k}_1 h) - 2\cos(\sqrt{2}\tilde{k}_2 h) - 4\cos(\tilde{k}_1 h/\sqrt{2})\cos(\tilde{k}_2 h/\sqrt{2}) &= 3\omega^2 h^2. \end{aligned} \tag{3.2}$$

Let us note that only the following values are admissible for different parameters:

$$\begin{aligned} 0 < k_1 h/\sqrt{2} < \pi \\ 0 < k_2 h/\sqrt{2} < \pi \\ -\pi < \tilde{k}_1 h/\sqrt{2} < \pi \\ 0 < \tilde{k}_2 h/\sqrt{2} < \pi \end{aligned}$$

because higher values cannot be represented by the computational grid. The condition for the $\tilde{k}_1$ is different than for others because a priori the parasitic wave could as well propagate to the left as to the right. In the present case we see that the only possible value for $\tilde{k}_1$ is $\tilde{k}_1 = k_1 - \sqrt{2}\pi/h$. Using this we get the following equation for $\tilde{k}_2$

$$\cos^2(\tilde{k}_2 h/\sqrt{2}) - \cos(k_1 h/\sqrt{2})\cos(\tilde{k}_2 h/\sqrt{2}) = \cos^2(k_2 h/\sqrt{2}) + \cos(k_1 h/\sqrt{2})\cos(k_2 h/\sqrt{2})$$

This has the solutions

$$\cos(\tilde{k}_2 h/\sqrt{2}) = \begin{cases} \cos(k_1 h/\sqrt{2}) + \cos(k_2 h/\sqrt{2}) \\ -\cos(k_2 h/\sqrt{2}) \end{cases} \tag{3.3}$$

The second solution is unacceptable because it gives a wave which is equal to the minus incident wave at the grid points (implying that the total field is zero...). As regards the first solution we note that when h tends to zero, it tends to two so that the corresponding $\tilde{k}_2$ must be imaginary. When this is the case we will use the notation $\tilde{k}_2 = i\kappa$ where $\kappa > 0$.

Then we have to take care of the boundary conditions. Using (3.1) and homogeneous Dirichlet conditions at $(0,0)$ and $(h/\sqrt{2}, h/\sqrt{2})$ give

$$\begin{aligned} 1 + R + \tilde{R} &= 0 \\ e^{-ik_2 h/\sqrt{2}} + Re^{ik_2 h/\sqrt{2}} - \tilde{R}e^{i\tilde{k}_2 h/\sqrt{2}} &= 0. \end{aligned} \tag{3.4}$$

Solving this we obtain

Proposition 3.1 *The reflection coefficient R is given by*

$$R = -\frac{e^{-ik_2 h/\sqrt{2}} + e^{i\tilde{k}_2 h/\sqrt{2}}}{e^{ik_2 h/\sqrt{2}} + e^{i\tilde{k}_2 h/\sqrt{2}}}.$$

When $h \to 0$ we have the asymptotic formula

$$R \simeq -1 + \frac{\sqrt{2}k_2 h\, i}{3-\sqrt{3}} + \frac{k_2^2 h^2}{(3-\sqrt{3})^2} + O(h^3).$$

In addition there exists h_0 such that for all $0 \le h \le h_0$, $|R| = 1$.

Proof To prove the asymptotic formula we note that when $\tilde{k}_2 = i\kappa$ the reflection coefficient becomes

$$R = -\frac{e^{-ik_2 h/\sqrt{2}} + e^{-\kappa h/\sqrt{2}}}{e^{ik_2 h/\sqrt{2}} + e^{-\kappa h/\sqrt{2}}} \tag{3.5}$$

whose absolute value is evidently exactly one. Then solving

$$\cosh(\kappa h/\sqrt{2}) = \cos(k_1 h/\sqrt{2}) + \cos(k_2 h/\sqrt{2})$$

gives

$$e^{-\kappa h/\sqrt{2}} = 2 - \sqrt{3} + ah^2 + O(h^4).$$

Then substituting this into (3.5) leads to the result. Note that the constant a does not have any effect.

■

We remark that when $\tilde{k}_2$ is not purely imaginary the absolute value of R is not one. Then let us take the Neumann conditions. Recall that when using the finite element method the Neumann condition is built into the variational formulation. When interpreting this back in terms of finite differences we get

$$\begin{aligned} 8u_{0,0} - 4u_{0,1} - 2u_{1/2,1/2} - 2u_{-1/2,1/2} &= 3\omega^2 h^2 u_{0,0} \\ 24u_{1/2,1/2} - 4u_{1/2,3/2} - 4u_{-1/2,1/2} - 4u_{0,1} & \\ -4u_{3/2,1/2} - 4u_{1,1} - 2u_{0,0} - 2u_{1,0} &= 9\omega^2 h^2 u_{1/2,1/2}. \end{aligned} \tag{3.6}$$

Then we have the following result.

Proposition 3.2 *The reflection coefficient R is given by*

$$R = -\frac{ww_1 e^{-ik_2h/\sqrt{2}} + w^* w_2 e^{i\tilde{k}_2 h/\sqrt{2}}}{ww_2 e^{i\tilde{k}_2 h/\sqrt{2}} + w^* w_1 e^{ik_2 h/\sqrt{2}}} \tag{3.7}$$

where $*$ denotes the complex conjugate and w, w_1 and w_2 are given by

$$\begin{aligned} w &= \sin^2(k_1h/\sqrt{2}) - \sin^2(k_2h/\sqrt{2}) + i\sin(k_2h/\sqrt{2})\big(2\cos(k_2h/\sqrt{2}) + \cos(k_1h/\sqrt{2})\big) \\ w_1 &= \sin^2(k_2h/\sqrt{2}) + e^{i\sqrt{2}\tilde{k}_2h} - \cos(k_1h/\sqrt{2})\big(e^{i\tilde{k}_2h/\sqrt{2}} + \cos(k_2h/\sqrt{2}) + \cos(k_1h/\sqrt{2})\big) \\ w_2 &= \cos(k_1h/\sqrt{2})\big(2e^{i\tilde{k}_2h/\sqrt{2}} + e^{-i\tilde{k}_2h/\sqrt{2}} + 3\cos(k_2h/\sqrt{2}) - \cos(k_1h/\sqrt{2})\big) + \\ & \quad 2 - 3\sin^2(k_2h/\sqrt{2}) - e^{i\sqrt{2}\tilde{k}_2h}. \end{aligned}$$

When $h \to 0$ we have the asymptotic formula

$$R \simeq 1 - i\,k_2h/\sqrt{2} + O(h^2)$$

and as before there exists h_0 such that for all $0 \le h \le h_0$, $|R| = 1$.

Proof First we note that as before $|R| = 1$ when $\tilde{k}_2$ is imaginary. Then putting (3.1) into (3.6) gives

$$\begin{gathered} c_1 + c_2R + c_3\tilde{R} = 0 \\ c_4e^{-ik_2h/\sqrt{2}} + c_5e^{ik_2h/\sqrt{2}}R - c_6e^{i\tilde{k}_2h/\sqrt{2}}\tilde{R} = 0 \end{gathered}$$

where c_i's are some constants that should be calculated. To this end we take the difference of the two sides of the first equation of (3.6) and then substitute $\exp(ik\cdot x)$ into it , which yields

$$\begin{aligned} 8 - 4e^{i\sqrt{2}k_2h} - 4\cos(k_1h/\sqrt{2})e^{ik_2h/\sqrt{2}} - 3\omega^2h^2 &= \\ 8 - 8\cos^2(k_2h/\sqrt{2}) + 4 - 8i\sin(k_2h/\sqrt{2})\cos(k_2h/\sqrt{2}) - & \\ 4\cos(k_1h/\sqrt{2})\cos(k_2h/\sqrt{2}) - 4i\cos(k_1h/\sqrt{2})\sin(k_2h/\sqrt{2}) - 12 + & \\ 4\cos^2(k_1h/\sqrt{2}) + 4\cos^2(k_2h/\sqrt{2}) + 4\cos(k_1h/\sqrt{2})\cos(k_2h/\sqrt{2}) &= -4w \end{aligned}$$

where (3.2) was used to get rid of ω. Evidently for the incident wave we just have to change the sign of the imaginary part of w, so that ignoring the factor -4 this shows that $c_1 = w^*$ and $c_2 = w$. Doing the same calculations with $\exp(i\tilde{k}\cdot x)$ gives

$$\begin{aligned} 8 - 4e^{i\sqrt{2}\tilde{k}_2 h} - 4\cos(\tilde{k}_1 h/\sqrt{2})e^{i\tilde{k}_2 h/\sqrt{2}} - 3\omega^2 h^2 &= \\ 8 - 4e^{i\sqrt{2}\tilde{k}_2 h} + 4\cos(k_1 h/\sqrt{2})e^{i\tilde{k}_2 h/\sqrt{2}} - 12 + & \\ 4\cos^2(k_1 h/\sqrt{2}) + 4\cos^2(k_2 h/\sqrt{2}) + 4\cos(k_1 h/\sqrt{2})\cos(k_2 h/\sqrt{2}) &= -4w_1. \end{aligned}$$

Using the second equation of (3.6) we verify that $c_4 = w$, $c_5 = w^*$ and $c_6 = w_2$, so that the equation to be solved becomes

$$\begin{aligned} w^* + wR + w_1\tilde{R} &= 0 \\ we^{-ik_2 h/\sqrt{2}} + w^* e^{ik_2 h/\sqrt{2}} R - w_2 e^{i\tilde{k}_2 h/\sqrt{2}} \tilde{R} &= 0. \end{aligned}$$

This gives the formula for R and then simple Taylor's expansion takes care of the rest.

■

3.2 Equilateral P_1 Element

Consider the situation in figure 3.2. We proceed as in the Q_1 case: first we recall the dispersion relation in the interior of the domain and then take into account the different boundary conditions. The equilateral P_1 elements correspond to the following difference scheme

$$12u_{m,n} - 2\sum_{l=1}^{6} u_l = 3\omega^2 h^2 u_{m,n}$$

where u_l are the six neighbors of $u_{m,n} = u(\sqrt{3}mh, nh)$ and the rules for m and n are as before. Substituting the plane wave the dispersion relation is found to be

$$16 - 8\cos^2(k_2 h/2) - 8\cos(\sqrt{3}k_1 h/2)\cos(k_2 h/2) = 3\omega^2 h^2. \quad (3.8)$$

Similar considerations as in the Q_1 case show that $\tilde{k}_1 = k_1 - 2\pi/\sqrt{3}h$, which leads to the following equation for $\tilde{k}_2$

$$\cos^2(\tilde{k}_2 h/2) - \cos(\sqrt{3}k_1 h/2)\cos(\tilde{k}_2 h/2) = \cos^2(k_2 h/2) + \cos(\sqrt{3}k_1 h/2)\cos(k_2 h/2)$$

with the solution

$$\cos(\tilde{k}_2 h/2) = \begin{cases} \cos(\sqrt{3}k_1 h/2) + \cos(k_2 h/2) \\ -\cos(k_2 h/2) \end{cases}.$$

Note that this is of the same form as (3.3), so that the same remarks apply: the second solution is not interesting and $\tilde{k}_2$ is imaginary when h is small.

Then homogeneous Dirichlet conditions at $(0,0)$ and $(\sqrt{3}h/2, h/2)$ lead to

$$\begin{aligned} 1+R+\tilde{R} &= 0 \\ e^{-ik_2h/2}+Re^{ik_2h/2}-\tilde{R}e^{i\tilde{k}_2h/2} &= 0. \end{aligned}$$

Solving this we obtain

Proposition 3.3 *The reflection coefficient R is given by*

$$R=-\frac{e^{-ik_2h/2}+e^{i\tilde{k}_2h/2}}{e^{ik_2h/2}+e^{i\tilde{k}_2h/2}}.$$

When $h\to 0$ the asymptotic formula is

$$R\simeq -1+\frac{k_2h\,i}{3-\sqrt{3}}+\frac{k_2^2h^2}{2(3-\sqrt{3})^2}+O(h^3)$$

and there exists h_0 such that for all $0\le h\le h_0$, $|R|=1$.

Proof As in the Q_1 case we find that

$$e^{-\kappa h/2}=2-\sqrt{3}+O(h^2).$$

Substituting this into the expression for R gives the result.

∎

Then the variational formulation gives us in case of the Neumann boundary conditions the following equations.

$$\begin{aligned} 4u_{0,0}-2u_{0,1}-u_{1/2,1/2}-u_{-1/2,1/2} &= \omega^2h^2u_{0,0} \\ 8u_{1/2,1/2}-2u_{1/2,3/2}-2u_{0,1}-2u_{1,1}-u_{0,0}-u_{1,0} &= 2\omega^2h^2u_{1/2,1/2} \end{aligned}$$

This then leads to

Proposition 3.4 *The reflection coefficient is given by*

$$R=-\frac{ww_1e^{-ik_2h/2}+w^*w_2e^{i\tilde{k}_2h/2}}{ww_2e^{i\tilde{k}_2h/2}+w^*w_1e^{ik_2h/2}}$$

where w, w_1 and w_2 are now

$$\begin{aligned} w &= 3i\sin(k_2h/2)\Big(2\cos(k_2h/2)+\cos(\sqrt{3}k_1h/2)\Big)+ \\ &\quad \cos(k_2h)-\cos(\sqrt{3}k_1h/2)\cos(k_2h/2) \\ w_1 &= -2\cos(k_2h)+3e^{i\tilde{k}_2h}-\cos(\sqrt{3}k_1h/2)\Big(3e^{i\tilde{k}_2h/2}+4\cos(k_2h/2)\Big) \\ w_2 &= 4\cos(k_2h)-3e^{i\tilde{k}_2h}+\cos(\sqrt{3}k_1h/2)\Big(6e^{i\tilde{k}_2h/2}+3e^{-i\tilde{k}_2h/2}+8\cos(k_2h/2)\Big). \end{aligned}$$

When $h\to 0$ we have the asymptotic formula

$$R\simeq 1-i\,k_2h/2+O(h^2)$$

and there exists h_0 such that for all $0\le h\le h_0$, $|R|=1$.

Proof We proceed exactly as in the Q_1 case. For instance let us calculate c_5 using the second equation of (3.2)

$$\begin{aligned}
24 - 6e^{ik_2h} - 12\cos(\sqrt{3}k_1h/2)e^{ik_2h/2} - 6\omega^2h^2 &= \\
24 - 12\cos^2(k_2h/2) + 6 - 12i\sin(k_2h/2)\cos(k_2h/2) - \\
12\cos(\sqrt{3}k_1h/2)\cos(k_2h/2) - 12i\cos(\sqrt{3}k_1h/2)\sin(k_2h/2) \\
-32 + 16\cos^2(k_2h/2) + 16\cos(\sqrt{3}k_1h/2)\cos(k_2h/2) &= 2w^*.
\end{aligned}$$

■

Note that the error term in P_1 case is obtained from the Q_1 case by dividing by $\sqrt{2}$ and this is also the ratio of the depths of the grooves.

3.3 Some Pictures

Let us illustrate the results we have obtained. First we take up the Dirichlet boundary condition; we plot the error of the reflection coefficient as a function of h and k_1 (note that because the absolute value of the correct reflection coefficient is one there is no difference between relative error and absolute error). Without the loss of generality we put $\omega = 1$ and $k_1 = \sin\alpha$ where $0 \leq \alpha \leq 90$ and α is the incidence angle. The error in the absolute value is mostly zero; only when $h \simeq 2.1$ it becomes non zero. In figures 3.3 and 3.4 one sees that the error grows linearly almost all the time and that it grows rather quickly. Consequently the linear term gives a good approximation of the error for the most values of h. Curiously the error is biggest at the normal incidence. We have plotted the P_1 case. The Q_1 case would give almost identical pictures except that the error is about $\sqrt{2}$ times bigger. Finally note that when h is big the variation looks 'random'.

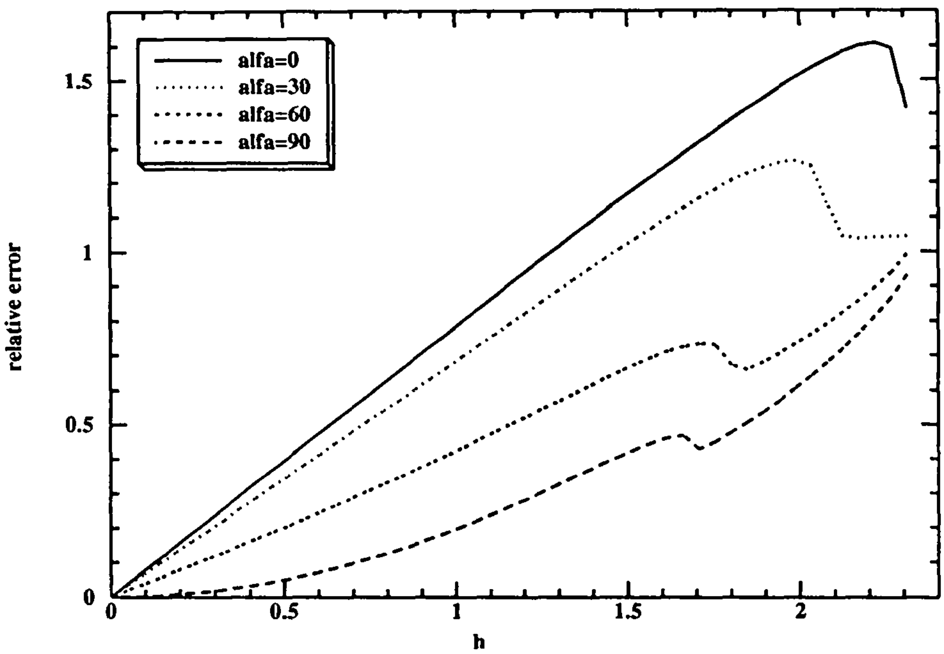

Figure 3.3: Error in the reflection: the Dirichlet case.

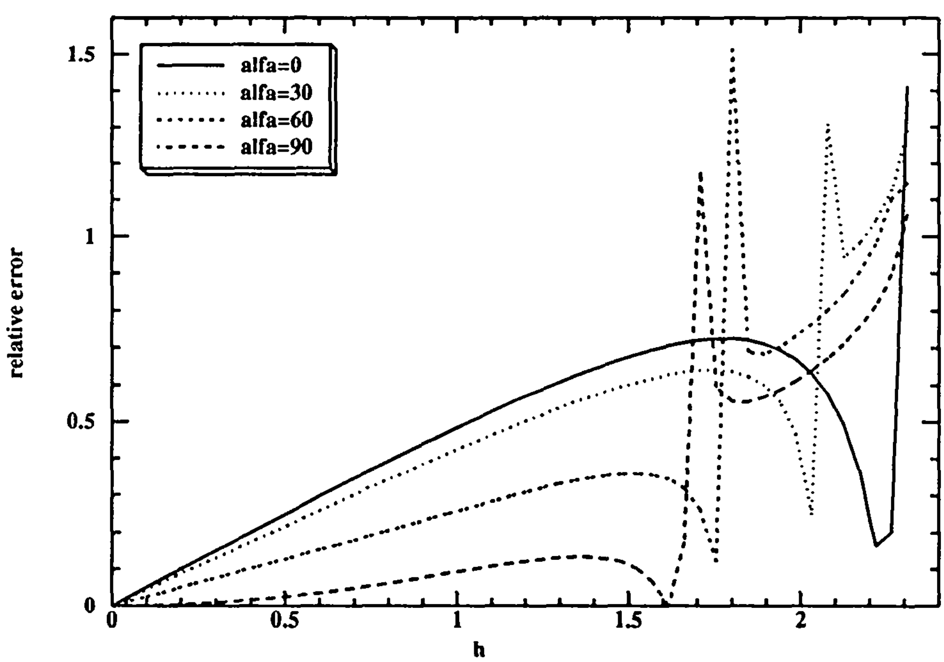

Figure 3.4: Error in the reflection: the Neumann case.

4 Reflection and Transmission

4.1 Some Preliminaries

When treating the transmission problem we will meet a simple singular perturbation problem whose solution we first develop. Consider the following linear system

$$A(\varepsilon)x = b(\varepsilon)$$

where $A \in \mathbf{C}^{n\times n}$ and x, $b \in \mathbf{C}^n$ and as usual ε is a small (real) parameter which is supposed to tend to zero. If $A(0)$ were invertible this would be trivial, but in our case (see below!) it happens that $A(0)$ is singular with rank $n-1$. However, $b(0)$ belongs to the image of $A(0)$ so we can hope to find a finite solution when $\varepsilon \longrightarrow 0$. We look for the solution in the form of the series and for definiteness we consider only the first three terms, so that the problem can be written as

$$(A_0 + A_1\varepsilon + A_2\varepsilon^2)(x_0 + x_1\varepsilon + x_2\varepsilon^2) = b_0 + b_1\varepsilon + b_2\varepsilon^2. \tag{4.1}$$

Denote by $R(A_0)$ the range of A_0 and by $N(A_0)$ its null space. By hypothesis the rank of A_0 is $n-1$ so that $N(A_0)$ is spanned by one vector; take any such vector and call it v_N. Also the orthogonal complement of $R(A_0)$ (denoted by $R(A_0)^\perp$) is one dimensional and we call its base vector v_R. From linear algebra we know that

$$x \in R(A)^\perp \iff x \in N(A^*)$$

where A^* is the transpose of A (or adjoint in the complex case). This gives us a simple way to calculate v_R. Now expanding (4.1) and putting the coefficients of the different powers of ε to zero we get

$$\begin{aligned} A_0x_0 &= b_0 \\ A_0x_1 + A_1x_0 &= b_1 \\ A_0x_2 + A_1x_1 + A_2x_0 &= b_2. \end{aligned}$$

By hypothesis $b_0 \in R(A_0)$ so that the first equation has an infinite number of solutions $x_0 = y_0 + \lambda_0 v_N$, where y_0 is some fixed solution and λ_0 is arbitrary. Then to have the solution to the next equation we must require that

$$b_1 - A_1x_0 \in R(A_0).$$

In other words $b_1 - A_1x_0$ is orthogonal to v_R which then gives immediately

$$\lambda_0 = \frac{(b_1 - A_1y_0, v_R)}{(A_1v_N, v_R)}.$$

Of course we must suppose that $(A_1 v_N, v_R) \neq 0$ which in fact turns out to be the case in the problem that interests us. Having calculated λ_0 we then solve

$$A_0 x_1 = b_1 - A_1 x_0$$

which gives $x_1 = y_1 + \lambda_1 v_N$. Then to ensure that the third equation has a solution we now demand that

$$b_2 - A_1 x_1 - A_2 x_0 \in R(A_0)$$

which then leads to

$$\lambda_1 = \frac{(b_2 - A_1 y_1 - A_2 x_0, v_R)}{(A_1 v_N, v_R)}.$$

Of course this process can be continued indefinitely.
More generally, suppose that the rank of A_0 is $n-k$, but still $b_0 \in R(A_0)$. Then both $N(A_0)$ and $R(A_0)^\perp$ are k dimensional, so that we look for the solution in the form

$$x_0 = y_0 + \sum_{i=1}^{k} \lambda_0^i v_N^i$$

where y_0 is as before some fixed solution and the vectors v_N^i form a base of $N(A_0)$. Similarly we can find the base vectors v_R^i in $R(A_0)^\perp$. Then simple calculations yield the following 'variational equation'.

$$\sum_{i=1}^{k} \lambda_0^i (A_1 v_N^i, v_R^j) = (b_1 - A_1 y_0, v_R^j) \quad \forall j = 1, \ldots, k.$$

To have a solution we must require that the matrix $M = \{ (A_1 v_N^i, v_R^j) \}_{i,j}$ is non singular and of course it would be convenient to have $(A_1 v_N^i, v_R^j) = 0$ when $i \neq j$. Naturally in general we may get neither the solution nor the orthogonality: this depends on the specific properties of A_0 and A_1. To calculate further terms we see that at every step in the algorithm one has to solve a system of the form $My = d$, where M is always the same but d changes. Since we will not need this general case we will not pursue it further.

4.2 Q_1 Element

Let us then take up the equation (2.3). When discretizing it with Q_1 elements we get in the interior of Ω_i the same dispersion relation as in (3.2) except that the left hand side has to be multiplied by c_i^2. Then analogously to the reflection case one looks for the solution in the following form.

$$u(x) = \begin{cases} \exp(i\bar{k} \cdot x) + R \exp(ik \cdot x) + \tilde{R} \exp(i\tilde{k} \cdot x) & \text{in } \Omega_1 \\ T \exp(i\xi \cdot x) + \tilde{T} \exp(i\tilde{\xi} \cdot x) & \text{in } \Omega_2 \end{cases} \tag{4.2}$$

where evidently $\xi = (\xi_1, -\xi_2)$ is the wave vector of the principal transmitted mode and $\tilde{\xi}$ is the wave vector of the parasitic mode. To get a solution with the same pseudoperiodicity properties as in the reflection case one must require that $\xi_1 = k_1$ and $\tilde{\xi}_1 = k_1 - \sqrt{2}\pi/h$. There are four free parameters because there are four conditions that this kind of solution must satisfy to be acceptable. Before stating these conditions we introduce some notations.

$$\begin{aligned} a_1 &= \cos(k_1 h/\sqrt{2}) \\ a_2 &= \exp(i\, k_2 h/\sqrt{2}) \\ a_3 &= \exp(i\, \tilde{k}_2 h/\sqrt{2}) \\ a_4 &= \exp(i\, \xi_2 h/\sqrt{2}) \\ a_5 &= \exp(i\, \tilde{\xi}_2 h/\sqrt{2}). \end{aligned}$$

Then for convenience let us recall that the given ω and k_1, the other parameters are obtained by the following sequence of equations

$$\begin{aligned} &3\omega^2 h^2 = 4c_1^2(3 - \cos^2(k_1 h/\sqrt{2}) - \cos^2(k_2 h/\sqrt{2}) - \cos(k_1 h/\sqrt{2})\cos(k_2 h/\sqrt{2})) \\ &\cos(\tilde{k}_2 h/\sqrt{2}) = \cos(k_1 h/\sqrt{2}) + \cos(k_2 h/\sqrt{2}) \\ &c_1^2(3 - \cos^2(k_1 h/\sqrt{2}) - \cos^2(k_2 h/\sqrt{2}) - \cos(k_1 h/\sqrt{2})\cos(k_2 h/\sqrt{2})) = \\ &c_2^2(3 - \cos^2(k_1 h/\sqrt{2}) - \cos^2(\xi_2 h/\sqrt{2}) - \cos(k_1 h/\sqrt{2})\cos(\xi_2 h/\sqrt{2})) \\ &\cos(\tilde{\xi}_2 h/\sqrt{2}) = \cos(k_1 h/\sqrt{2}) + \cos(\xi_2 h/\sqrt{2}). \end{aligned}$$

First of all we must require the continuity at the interface which gives the following two equations

$$\begin{aligned} 1 + R + \tilde{R} &= T + \tilde{T} \\ 1/a_2 + a_2 R - a_3 \tilde{R} &= T/a_4 - \tilde{T}/a_5. \end{aligned}$$

The two other equations come from the variational formulation. Let us first write down the difference equations.

$$\begin{aligned} &(24\mu_2 + 8\mu_1)u_{0,0} - 4\mu_1 u_{0,1} - 2(\mu_1 + \mu_2)(u_{1/2,1/2} + u_{-1/2,1/2}) \\ &\qquad -4\mu_2(u_{-1,0} + u_{-1/2,1/2} + u_{0,-1} + u_{1/2,-1/2} + u_{1,0}) = (9\rho_2 + 3\rho_1)\omega^2 h^2 u_{0,0} \\ &(24\mu_1 + 8\mu_2)u_{1/2,1/2} - 4\mu_2 u_{1/2,-1/2} - 2(\mu_1 + \mu_2)(u_{0,0} + u_{1,0}) \\ &\qquad -4\mu_1(u_{-1/2,1/2} + u_{0,1} + u_{1/2,3/2} + u_{1,1} + u_{3/2,1/2}) = (9\rho_1 + 3\rho_2)\omega^2 h^2 u_{1/2,1/2}. \end{aligned}$$

Then substituting (4.2) into the above equation yield

$$\begin{aligned} &(24\mu_2 + 8\mu_1)(T + \tilde{T}) - 4\mu_1(1/a_2^2 + a_2^2 R + a_3^2 \tilde{R}) \\ &-4a_1(\mu_1 + \mu_2)(T/a_4 - \tilde{T}/a_5) - 8\mu_2(2a_1^2 - 1)(T + \tilde{T}) \\ &\qquad -8\mu_2 a_1(a_4 T - a_5 \tilde{T}) - 4\mu_2(a_4^2 T + a_5^2 \tilde{T}) = (9\rho_2 + 3\rho_1)\omega^2 h^2 (T + \tilde{T}) \end{aligned}$$

$$
\begin{aligned}
&(24\mu_1 + 8\mu_2)(T/a_4 - \tilde{T}/a_5) - 4\mu_1(1/a_2^3 + a_2^3 R - a_3^3 \tilde{R}) \\
&\quad -4a_1(\mu_1 + \mu_2)(T + \tilde{T}) - 8\mu_1(2a_1^2 - 1)(T/a_4 - \tilde{T}/a_5) \\
&\qquad -8\mu_1 a_1(1/a_2^2 + a_2^2 R + a_3^2 \tilde{R}) - 4\mu_2(a_4 T - a_5 \tilde{T}) \;=\; (9\rho_1 + 3\rho_2)\omega^2 h^2 (T/a_4 - \tilde{T}/a_5).
\end{aligned}
$$

So all in all we have a system of four linear equations. Now it would be too complicated to try write down the exact solution (or to calculate it!) so we try to find directly the asymptotic solution. To this end we first note the following expansions.

$$
\begin{aligned}
\omega^2 &= c_1^2 |k|^2 + O(h^2) \\
a_1 &= 1 - k_1^2 h^2/4 + O(h^4) \\
a_2 &= 1 + i\, k_2 h/\sqrt{2} - k_2^2 h^2/4 + O(h^3) \\
a_3 &= 2 - \sqrt{3} + (2\sqrt{3} - 3)\, |k|^2 h/12 + O(h^3) \\
a_4 &= 1 + i\, |k|\, h\sqrt{c_1^2 - c_2^2 \sin^2\alpha}/c_2\sqrt{2} - |k|^2 h^2 (c_1^2 - c_2^2 \sin^2\alpha)/4c_2^2 + O(h^3) \\
a_5 &= 2 - \sqrt{3} + (2\sqrt{3} - 3) c_1^2 |k|^2 h^2/12 c_2^2 + O(h^3)
\end{aligned}
$$

where α is the angle of incidence and so we can also write $k_1 = |k| \sin\alpha$ and $k_2 = |k| \cos\alpha$. Substituting these expansions into the equations and putting all into a matrix form as in the preceding section we then find that

$$
A_0 = \begin{pmatrix}
1 & 1 & -1 & -1 \\
1 & \sqrt{3} - 2 & -1 & 1/(2 - \sqrt{3}) \\
-4c_1 z_1 & 4(4\sqrt{3} - 7)c_1 z_1 & 4c_1 z_1 & \dfrac{(20 - 8\sqrt{3})c_1 z_1 + 12(\sqrt{3} - 1)c_2 z_2}{2 - \sqrt{3}} \\
-12 c_1 z_1 & 4(12 - 7\sqrt{3})c_1 z_1 & 12 c_1 z_1 & \dfrac{(4\sqrt{3} - 24)c_1 z_1 - 12(\sqrt{3} - 1)c_2 z_2}{2 - \sqrt{3}}
\end{pmatrix}.
$$

The unknown vector is $x = (R, \tilde{R}, T, \tilde{T})$. The rank of A_0 is three and we easily calculate that $v_N = (1, 0, 1, 0)$ and $v_R = (4c_1 z_1, 12 c_1 z_1, 1, 1)$. As regards the right hand side we obtain

$$
b = \begin{pmatrix} -1 \\ -1 \\ 4c_1 z_1 \\ 12 c_1 z_1 \end{pmatrix} + \frac{i\ |k|\, h \cos\alpha}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -8c_1 z_1 \\ -28 c_1 z_1 \end{pmatrix} + \frac{|k|^2 h^2}{4} \begin{pmatrix} 0 \\ \cos^2\alpha \\ -16 c_1 z_1 \cos^2\alpha \\ -4c_1 z_1 (17 - 15 \sin^2\alpha) \end{pmatrix}
$$

which we write as $b = b_0 + i\, b_1 h + b_2 h^2$, so that all the vectors b_i are real. Similarly it turns out that we can write $A = A_0 + i\, A_1 h + A_2 h^2$ where A_i are real matrices. Now we use the algorithm of the previous section: $b_0 \in R(A_0)$ so we first calculate one solution to $A_0 x_0 = b_0$ which is found to be $y_0 = (-1, 0, 0, 0)$. Then we need A_1 which is given by

$$A_1 = \frac{|k|}{c_2\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_2\cos\alpha & 0 & \sqrt{c_1^2 - c_2^2\sin^2\alpha} & 0 \\ -8c_1c_2z_1\cos\alpha & 0 & (4c_1z_1 - 12c_2z_2)\sqrt{c_1^2 - c_2^2\sin^2\alpha} & 0 \\ -28c_1z_1\cos\alpha & 0 & -(16c_1z_1 + 12c_2z_2)\sqrt{c_1^2 - c_2^2\sin^2\alpha} & 0 \end{pmatrix}.$$

Next we calculate that

$$\begin{aligned} \lambda_0 &= \frac{(b_1 - A_1y_0, v_R)}{(A_1v_N, v_R)} \\ &= \frac{2c_1z_1\cos\alpha}{c_1z_1\cos\alpha + z_2\sqrt{c_1^2 - c_2^2\sin^2\alpha}}. \end{aligned}$$

This gives us already that

$$\begin{aligned} R &= \frac{c_1z_1\cos\alpha - z_2\sqrt{c_1^2 - c_2^2\sin^2\alpha}}{c_1z_1\cos\alpha + z_2\sqrt{c_1^2 - c_2^2\sin^2\alpha}} + O(h) \\ T &= \frac{2c_1z_1\cos\alpha}{c_1z_1\cos\alpha + z_2\sqrt{c_1^2 - c_2^2\sin^2\alpha}} + O(h) \\ \tilde{R} &= O(h) \\ \tilde{T} &= O(h). \end{aligned}$$

We call the zero order terms of R and T r_0 and t_0. Note that r_0 and t_0 are the 'right' coefficients given in (2.4). This is rather natural because after all our boundary 'converges' in some sense to a straight line. We still want to calculate the $O(h)$ term of R and T. Here the intermediate expressions are already quite long so that we only state the final result. With the notation $R = r_0 + i\,hr_1 + O(h^2)$ and $T = t_0 + i\,ht_1 + O(h^2)$ we obtain

Theorem 4.1 *The first order terms r_1 and t_1 are given by*

$$\begin{aligned} r_1 &= \frac{\sqrt{2}\,|k|\cos\alpha}{6c_2(c_1z_1 + c_2z_2)(c_1z_1\cos\alpha + z_2\sqrt{c_1^2 - c_2^2\sin^2\alpha})^2}\Big(\sqrt{3}c_1^4z_1^2z_2 - 3c_1^3c_2z_1^3\cos^2\alpha \\ &\quad +(3 - 2\sqrt{3})c_1^3c_2z_1z_2^2 - (3\cos^2\alpha - \sqrt{3}\sin^2\alpha)c_1^2c_2^2z_1^2z_2 + \\ &\quad (2\sqrt{3} - 3)c_1c_2^3z_1z_2^2\sin^2\alpha + (3 + \sqrt{3})c_1^2c_2^2z_2^3 - (3 + \sqrt{3})c_2^4z_2^3\sin^2\alpha\Big) \\ t_1 &= \frac{\sqrt{2}c_1z_1\,|k|\cos\alpha}{6c_2(c_1z_1 + c_2z_2)(c_1z_1\cos\alpha + z_2\sqrt{c_1^2 - c_2^2\sin^2\alpha})^2} \end{aligned}$$

$$\Big(3(c_1z_1+c_2z_2)(c_1^2z_2-c_2^2z_2\sin^2\alpha-c_1c_2z_1\cos^2\alpha)+$$
$$\cos\alpha(c_1z_1-c_2z_2)\Big((3-\sqrt{3})c_1z_1+(3+\sqrt{3})c_2z_2\Big)\sqrt{c_1^2-c_2^2\sin^2\alpha}\Big).$$

Note in particular that putting $r_1=r_1(c_1,c_2,z_1,z_2)$ and $t_1=t_1(c_1,c_2,z_1,z_2)$ we have

$$\begin{aligned} r_1(s_1c_1,s_1c_2,s_2z_1,s_2z_2) &= r_1(c_1,c_2,z_1,z_2) \\ t_1(s_1c_1,s_1c_2,s_2z_1,s_2z_2) &= t_1(c_1,c_2,z_1,z_2). \end{aligned} \tag{4.3}$$

This means that when analysing the error term there are only two parameters: c_2/c_1 and z_2/z_1.

4.3 Equilateral P_1 Element

We proceed exactly as in the preceding section. The parameters a_i now are

$$\begin{aligned} a_1 &= \cos(\sqrt{3}k_1h/2) \\ a_2 &= \exp(i\,k_2h/2) \\ a_3 &= \exp(i\,\tilde{k}_2h/2) \\ a_4 &= \exp(i\,\xi_2h/2) \\ a_5 &= \exp(i\,\tilde{\xi}_2h/2) \end{aligned}$$

and the different components of the wave vectors are calculated from the following equations

$$\begin{aligned} &3\omega^2h^2=8c_1^2(2-\cos(k_2h/2)^2-\cos(\sqrt{3}k_1h/2)\cos(k_2h/2)) \\ &\cos(\tilde{k}_2h/2)=\cos(\sqrt{3}k_1h/2)+\cos(k_2h/2) \\ &c_1^2(2-\cos(k_2h/2)^2-\cos(\sqrt{3}k_1h/2)\cos(k_2h/2))= \\ &c_2^2(2-\cos(\xi_2h/2)^2-\cos(\sqrt{3}k_1h/2)\cos(\xi_2h/2)) \\ &\cos(\tilde{\xi}_2h/2)=\cos(\sqrt{3}k_1h/2)+\cos(\xi_2h/2). \end{aligned}$$

With these notations the first two equations are formally exactly the same as in the Q_1 case, that is we have

$$\begin{aligned} 1+R+\tilde{R} &= T+\tilde{T} \\ 1/a_2+a_2R-a_3\tilde{R} &= T/a_4-\tilde{T}/a_5. \end{aligned}$$

The variational formulation at the interface gives

$$\begin{aligned} &(8\mu_2+4\mu_1)u_{0,0}-2\mu_1u_{0,1}-(\mu_1+\mu_2)(u_{1/2,1/2}+u_{-1/2,1/2}) \\ &\qquad -2\mu_2(u_{-1/2,1/2}+u_{0,-1}+u_{1/2,-1/2}) = (2\rho_2+\rho_1)\omega^2h^2u_{0,0} \\ &(8\mu_1+4\mu_2)u_{1/2,1/2}-2\mu_2u_{1/2,-1/2}-(\mu_1+\mu_2)(u_{0,0}+u_{1,0}) \\ &\qquad -2\mu_1(u_{0,1}+u_{1/2,3/2}+u_{1,1}) = (2\rho_1+\rho_2)\omega^2h^2u_{1/2,1/2}. \end{aligned}$$

Substituting (4.2) into the above equation yield

$$\begin{aligned}
(8\mu_2+4\mu_1)(T+\tilde{T})-2\mu_1(1/a_2^2+a_2^2R+a_3^2\tilde{R}) & \\
-2a_1(\mu_1+\mu_2)(T/a_4-\tilde{T}/a_5)-4\mu_2a_1(a_4T-a_5\tilde{T}) & \\
-2\mu_2(a_4^2T+a_5^2\tilde{T}) &= (2\rho_2+\rho_1)\omega^2h^2(T+\tilde{T}) \\
(8\mu_1+4\mu_2)(T/a_4-\tilde{T}/a_5)-2\mu_1(1/a_2^3+a_2^3R-a_3^3\tilde{R}) & \\
-2a_1(\mu_1+\mu_2)(T+\tilde{T})-4\mu_1a_1(1/a_2^2+a_2^2R+a_3^2\tilde{R}) & \\
-2\mu_2(a_4T-a_5\tilde{T}) &= (2\rho_1+\rho_2)\omega^2h^2(T/a_4-\tilde{T}/a_5).
\end{aligned}$$

Then expanding the parameters gives

$$\begin{aligned}
\omega^2 &= c_1^2\,|k|^2+O(h^2) \\
a_1 &= 1-3k_1^2h^2/8+O(h^4) \\
a_2 &= 1+i\,k_2h/2-k_2^2h^2/8+O(h^3) \\
a_3 &= 2-\sqrt{3}+(2\sqrt{3}-3)(2\sin^2\alpha+1)\,|k|^2\,h/24+O(h^3) \\
a_4 &= 1+i\,|k|\,h\sqrt{c_1^2-c_2^2\sin^2\alpha}/(2c_2)-|k|^2\,h^2(c_1^2-c_2^2\sin^2\alpha)/(8c_2^2)+O(h^3) \\
a_5 &= 2-\sqrt{3}+(2\sqrt{3}-3)(c_1^2+2c_2^2\sin^2\alpha)\,|k|^2\,h^2/(24c_2^2)+O(h^3).
\end{aligned}$$

This leads to

$$A_0=\begin{pmatrix}
1 & 1 & -1 & -1 \\
1 & \sqrt{3}-2 & -1 & 1/(2-\sqrt{3}) \\
-2c_1z_1 & (8\sqrt{3}-14)c_1z_1 & 2c_1z_1 & \dfrac{(10-4\sqrt{3})c_1z_1+6(\sqrt{3}-1)c_2z_2}{2-\sqrt{3}} \\
-6c_1z_1 & (24-14\sqrt{3})c_1z_1 & 6c_1z_1 & \dfrac{(2\sqrt{3}-12)c_1z_1-12(\sqrt{3}-1)c_2z_2}{2-\sqrt{3}}
\end{pmatrix}.$$

Next simple calculations show that $v_N=(1,0,1,0)$ and $v_R=(2c_1z_1,6c_1z_1,1,1)$ and the right hand side is

$$b=\begin{pmatrix} -1 \\ -1 \\ 2c_1z_1 \\ 6c_1z_1 \end{pmatrix}+\frac{i\,|k|\,h\cos\alpha}{2}\begin{pmatrix} 0 \\ 1 \\ -4c_1z_1 \\ -14c_1z_1 \end{pmatrix}+\frac{|k|^2\,h^2}{8}\begin{pmatrix} 0 \\ \cos^2\alpha \\ -8c_1z_1\cos^2\alpha \\ 2c_1z_1(11\sin^2\alpha-17) \end{pmatrix}.$$

As before, when solving $A_0x_0=b_0$ we obtain $y_0=(-1,0,0,0)$. Then we need A_1 which is given by

$$A_1=\frac{|k|}{2c_2}\begin{pmatrix}
0 & 0 & 0 & 0 \\
c_2\cos\alpha & 0 & \sqrt{c_1^2-c_2^2\sin^2\alpha} & 0 \\
-4c_1c_2z_1\cos\alpha & 0 & (2c_1z_1-6c_2z_2)\sqrt{c_1^2-c_2^2\sin^2\alpha} & 0 \\
-14c_1z_1\cos\alpha & 0 & -(8c_1z_1+6c_2z_2)\sqrt{c_1^2-c_2^2\sin^2\alpha} & 0
\end{pmatrix}.$$

Then solving for λ_0 and x_0 we notice that the zero order terms are exactly the same as in the Q_1 case, which is rather natural. Then for details of the computation of the first order terms we refer to the appendix and here we only state the result. Note that this computation is almost identical to the computation that lead to the theorem 4.1.

Theorem 4.2 *The first order terms of R and T are obtained by dividing r_1 and t_1 of the Q_1 case by $\sqrt{2}$.*

4.4 Five Points Scheme

Rotating 45 degrees the five points scheme gives the following method.

$$4u_{m,n} - u_{m+1/2,n+1/2} - u_{m+1/2,n-1/2} - u_{m-1/2,n+1/2} - u_{m-1/2,n-1/2} = \omega^2 h^2 u_{m,n}$$

where $u_{m,n}$ are like in the Q_1 case. This has the dispersion relation

$$4 - 4\cos(k_1 h/\sqrt{2})\cos(k_2 h/\sqrt{2}) = \omega^2 h^2.$$

Then putting $\tilde{k}_1 = k_1 - \sqrt{2}\pi/h$ and trying to solve $\tilde{k}_2$ one obtains the 'impossible' equation

$$\cos(\tilde{k}_2 h/\sqrt{2}) = -\cos(k_2 h/\sqrt{2}).$$

We can never find a solution even if one allows complex values for $\tilde{k}_2$. Comparing to (3.3) we see that it is as if we had 'lost the good solution'.

To solve this apparent paradox, we first note that in figure 4.1 there is no direct connection between the point A and the point above it so that it is unnecessary to impose the exact continuity at the interface and so we do not 'need' a parasitic wave! This is why we did not treat the reflection problem for the five points scheme. However, this does not mean that the reflection and transmission coefficients are exact, as we shall see.

So let us simply look for the solution of the following form.

$$\begin{aligned} u(x_1, x_2) &= \exp(i\,\bar{k}\cdot x) + R\exp(i\,k\cdot x) && x_2 \geq h/\sqrt{2} \\ u(x_1, x_2) &= T\exp(i\,\xi\cdot x) && x_2 \leq 0. \end{aligned}$$

Variational formulation gives at the interface

$$\begin{aligned} (18\mu_2 + 6\mu_1)u_{0,0} - 3(\mu_1+\mu_2)(u_{1/2,1/2} + u_{-1/2,1/2}) & \\ -6\mu_2(u_{-1/2,1/2} + u_{1/2,-1/2}) &= (5\rho_2 + \rho_1)\omega^2 h^2 u_{0,0} \\ (18\mu_1 + 6\mu_2)u_{1/2,1/2} - 3(\mu_1+\mu_2)(u_{0,0} + u_{1,0}) & \\ -6\mu_1(u_{0,1} + u_{1,1}) &= (5\rho_1 + \rho_2)\omega^2 h^2 u_{1/2,1/2}. \end{aligned}$$

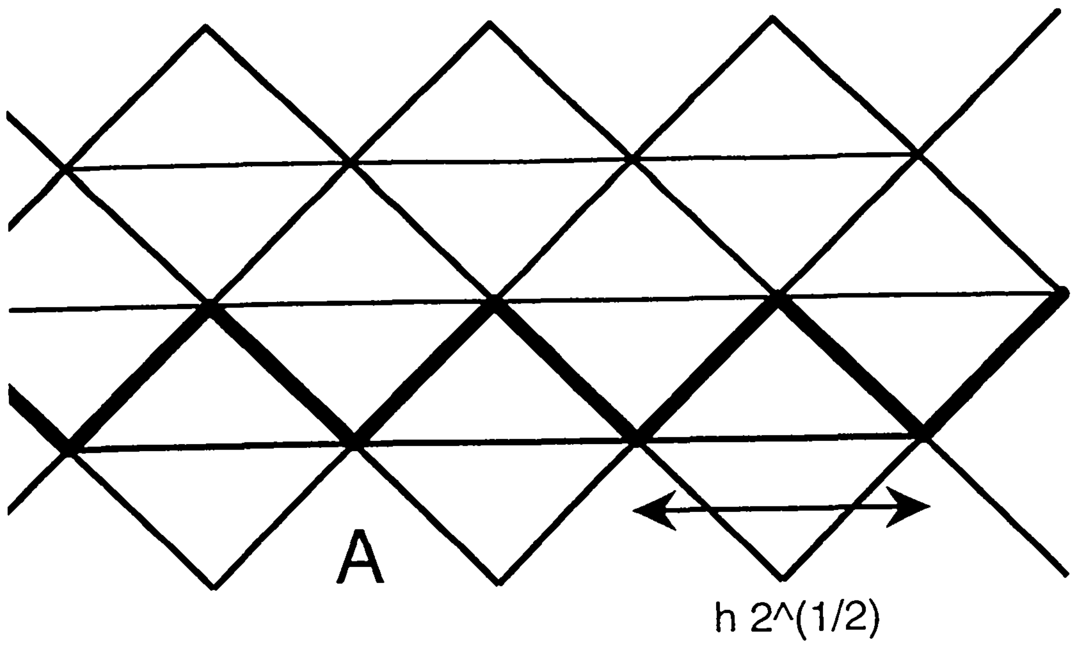

Figure 4.1: Grid for the five points scheme.

Note that interpreting the underlying triangulation differently we could replace $5\rho_i + \rho_j$ by $4\rho_i + 2\rho_j$. Then substituting the proposed solution into the above equation yield

$$\begin{aligned}
(18\mu_2 + 6\mu_1)T - 12\mu_2 a_1 a_4 T & \\
-6a_1(\mu_1 + \mu_2)(1/a_2 + Ra_2) &= (5\rho_2 + \rho_1)\omega^2 h^2 T \\
(18\mu_1 + 6\mu_2)(1/a_2 + Ra_2) - 6a_1(\mu_1 + \mu_2)T & \\
-12\mu_1 a_1(1/a_2^2 + a_2^2 R) &= (5\rho_1 + \rho_2)\omega^2 h^2 (1/a_2 + Ra_2)
\end{aligned}$$

where we have used as before the following notations.

$$\begin{aligned}
a_1 &= \cos(k_1 h/\sqrt{2}) \\
a_2 &= \exp(i\, k_2 h/\sqrt{2}) \\
a_4 &= \exp(i\, \xi_2 h/\sqrt{2}).
\end{aligned}$$

Even with these two simple equations the exact solution is so complicated that we do not write it down. Calculating as before the asymptotic expansion we find the following result.

Theorem 4.3 *The coefficients r_1 and t_1 are given by*

$$\begin{aligned}
r_1 = & \frac{\sqrt{2}\,|k|\cos\alpha}{2c_2(c_1 z_1 + c_2 z_2)(c_1 z_1 \cos\alpha + z_2\sqrt{c_1^2 - c_2^2\sin^2\alpha})^2} \\
& \Big(c_1^4 z_1^2 z_2 - c_1^3 c_2 z_1^3 \cos^2\alpha - c_1^3 c_2 z_1 z_2^2 - c_1^2 c_2^2 z_1^2 z_2 + \\
& c_1 c_2^3 z_1 z_2^2 \sin^2\alpha + 2c_1^2 c_2^2 z_2^3 - 2c_2^4 z_2^3 \sin^2\alpha\Big)
\end{aligned}$$

$$
\begin{aligned}
t_1 \quad = \quad & \frac{\sqrt{2}c_1 z_1 \, |k| \cos\alpha}{2c_2(c_1 z_1 + c_2 z_2)(c_1 z_1 \cos\alpha + z_2\sqrt{c_1^2 - c_2^2 \sin^2\alpha})^2} \\
& \Big((c_1 z_1 + c_2 z_2)(c_1^2 z_2 - c_2^2 z_2 \sin^2\alpha - c_1 c_2 z_1 \cos^2\alpha) + \\
& 2c_2 z_2 \cos\alpha (c_1 z_1 - c_2 z_2)\sqrt{c_1^2 - c_2^2 \sin^2\alpha}\Big).
\end{aligned}
$$

Note that choosing $5\rho_i + \rho_j$ rather than $4\rho_i + 2\rho_j$ does not have any effect on these coefficients (this choice has influence on the second order terms).

4.5 Total Reflection

We recall that taking Ω_1 to be the upper half plane in (2.3) there is a total reflection if $c_1 \leq c_2 \sin\alpha$, that is there are no propagating transmitted waves. In some sense all the energy is reflected back because $|R| = 1$. We shall analyse below how the three schemes we have been considering behave with respect to this phenomenon. First we will verify that also numerically we have $|R| = 1$, and then calculate the error in the critical angle.

Of course we must first introduce some notations. Let $A \in \mathbb{C}^{n\times n}$ and $x,\ b \in \mathbb{C}^n$. Then we denote by A_l the matrix which is obtained from A by removing the l'th column which is denoted by a^l. Then we define $\det(b, A_l)$ to be the determinant of the matrix A with the l'th column replaced by b. Now we can state the Kramer's rule: the solution of $Ax = b$ is

$$x_l = \det(b, A_l)/\det(A)$$

where x_l is obviously the l'th component of x. Then we obtain

Theorem 4.4 *$|R| = 1$ for the Q_1 scheme (respectively equilateral P_1 and five points scheme) if the condition D_1 (respectively D_2 and D_3) below is verified*

$$
\begin{array}{lll}
D_1 & \frac{c_1^2}{c_2^2} \leq & \frac{2 - \cos^2(k_1 h/\sqrt{2}) - \cos(k_1 h/\sqrt{2})}{3 - \cos^2(k_1 h/\sqrt{2}) - \cos(k_2 h/\sqrt{2})^2 - \cos(k_1 h/\sqrt{2})\cos(k_2 h/\sqrt{2})} \\
D_2 & \frac{c_1^2}{c_2^2} \leq & \frac{1 - \cos(\sqrt{3}k_1 h/2)}{2 - \cos(k_2 h/2)^2 - \cos(\sqrt{3}k_1 h/2)\cos(k_2 h/2)} \\
D_3 & \frac{c_1^2}{c_2^2} \leq & \frac{1 - \cos(k_1 h/\sqrt{2})}{1 - \cos(k_1 h/\sqrt{2})\cos(k_2 h/\sqrt{2})}
\end{array}
.
$$

Proof Let us prove the equilateral P_1 case. Put $\cos(\sqrt{3}k_1 h/2) = a$, $\cos(k_2 h/2) = d$ and $\cos(\xi_2 h/2) = y$. Then the equation for ξ_2 can be written as

$$c_1^2(2 - ad - d^2) = c_2^2(2 - ay - y^2)$$

If the positive solution of y is bigger than one the transmitted wave is evanescent, and a little algebra shows that the condition D_2 is exactly $y \geq 1$. Then we have to show that this implies that $|R| = 1$. Now writing our four equations for the reflection and the transmission coefficients in the matrix form $Ax = b$ we see that using Kramer's rule $R = \det(b, A_1)/\det(A)$. Then we verify that $b^* = a^1$ always and that the elements of A_1 are all real when the condition D_2 is satisfied. Then the elementary properties of the determinant imply that $R = z^*/z$ where z is some complex number and the conclusion follows. The proof of the other cases is similar.

■

The critical angle in the continuous case is $\alpha = \arcsin(c_1/c_2)$. For the different schemes this can be calculated by replacing the inequality sign in the conditions D_i by the equality sign. This yields

Proposition 4.1 *The critical angles α_1 for Q_1 scheme, α_2 for equilateral P_1 scheme and α_3 for the five points scheme are given by*

$$\begin{aligned}
\alpha_1 &= \arcsin(c_1/c_2) + \frac{c_1(c_1^2 - c_2^2)(4c_1^2 - 3c_2^2)|k|^2h^2}{48c_2^5\sqrt{1 - c_1^2/c_2^2}} + O(h^4) \\
\alpha_2 &= \arcsin(c_1/c_2) + \frac{c_1(c_1^2 - c_2^2)|k|^2h^2}{32c_2^3\sqrt{1 - c_1^2/c_2^2}} + O(h^4) \\
\alpha_3 &= \arcsin(c_1/c_2) + \frac{c_1(c_1^2 - c_2^2)(4c_1^2 + c_2^2)|k|^2h^2}{48c_2^5\sqrt{1 - c_1^2/c_2^2}} + O(h^4).
\end{aligned}$$

Proof These are simply obtained by expanding the conditions D_i.

■

As a simple consequence we see that α_1 is the most accurate when $\sqrt{3/8} < c_1/c_2 < 1$, α_2 when $1/\sqrt{8} < c_1/c_2 < \sqrt{3/8}$ and α_3 when $0 < c_1/c_2 < 1/\sqrt{8}$.

4.6 Some Comparisons

Let us first analyse the first order error terms. It is sufficient to consider the five points scheme and the P_1 case because the Q_1 terms are obtained from the P_1 terms by multiplying by $\sqrt{2}$. Recalling the property (4.3) we have then three independent parameters: c_2/c_1, z_2/z_1 and α. In figures 4.2, 4.3, 4.4 and 4.5 we have plotted the first order terms as a function of z_2/z_1 for different values of parameters. It is seen that for the most values error terms are bigger for the five points scheme. Note that the absolute values of the 'right' coefficients (given in (2.4)) are of the order unity (or smaller) so that the error terms are quite big in the two cases.

All the schemes considered are of the second order; however, in all cases the discrete boundary introduces error terms of the first order. This suggests that in scattering

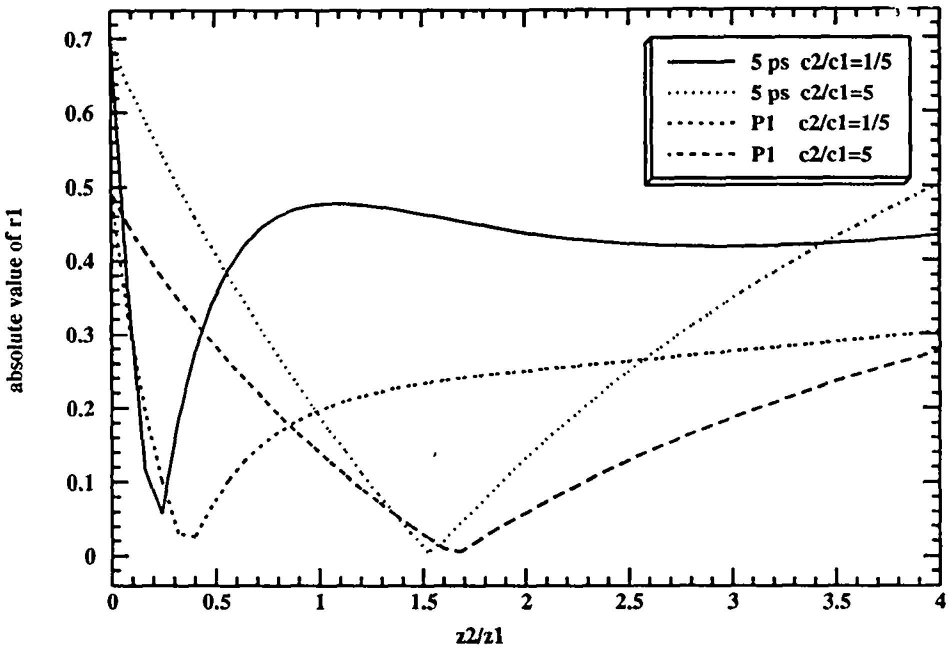

Figure 4.2: First order reflection term; $\alpha = 10$.

problems one should discretize very carefully the surface of the obstacle: using simply the uniform grid everywhere which transforms a smooth surface to a kind of staircase creates quite large errors. In fact Mittra in [MI] has arrived experimentally to a same conclusion: in his numerical simulations the errors due to the staircase discretization were too big to be acceptable.

Then in figures 4.6, 4.7, 4.8 and 4.9 we have plotted the reflection and transmission coefficients in the complex plane as a function of h, when h varies from zero to $\pi/2$. When $\pi/2 \leq h \leq \pi$ the variation of the coefficients is rather chaotic and they do not bear much relation to 'right' coefficients (or to each other). Again it is plain that the errors are quite big.

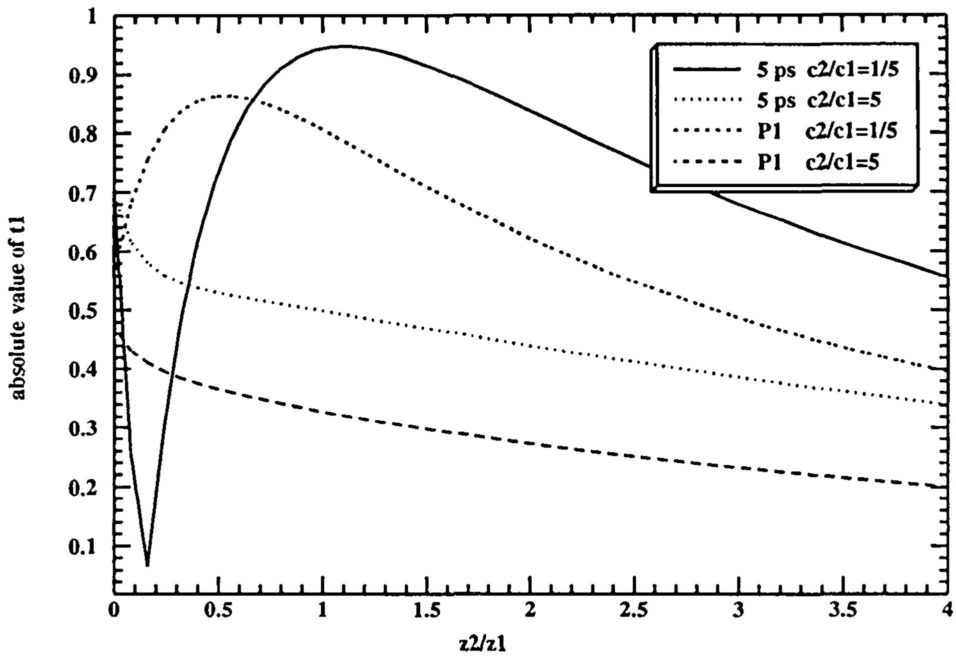

Figure 4.3: First order transmission term; $\alpha = 10$.

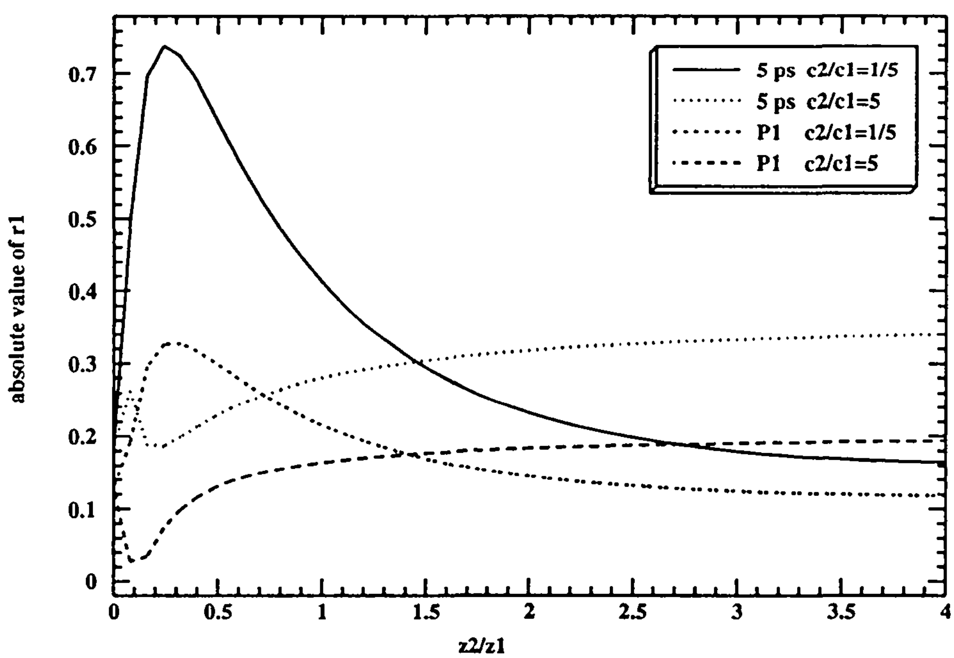

Figure 4.4: First order reflection term; $\alpha = 75$.

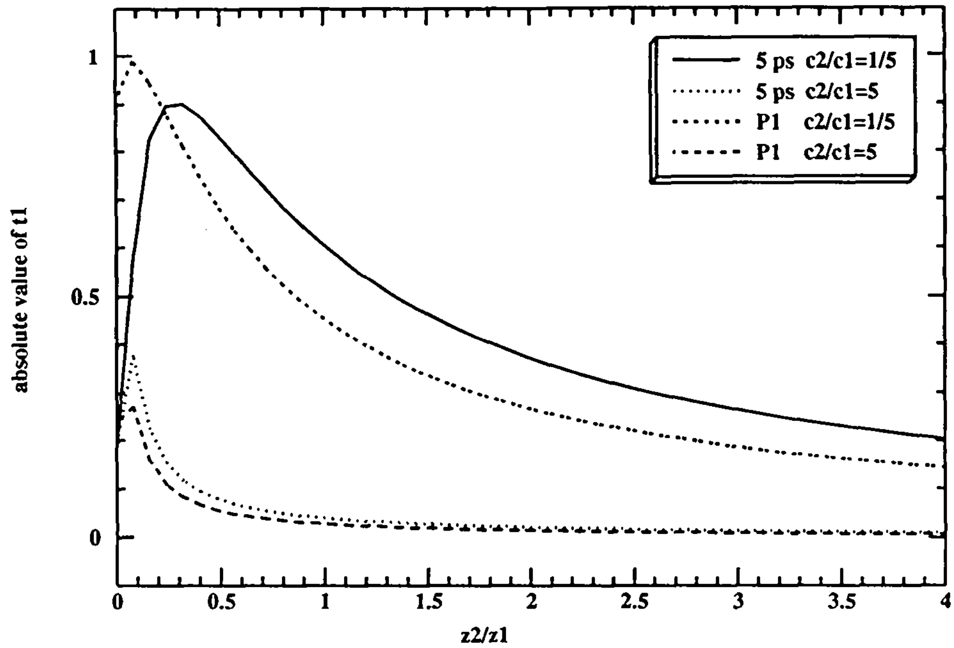

Figure 4.5: First order transmission term; $\alpha = 75$.

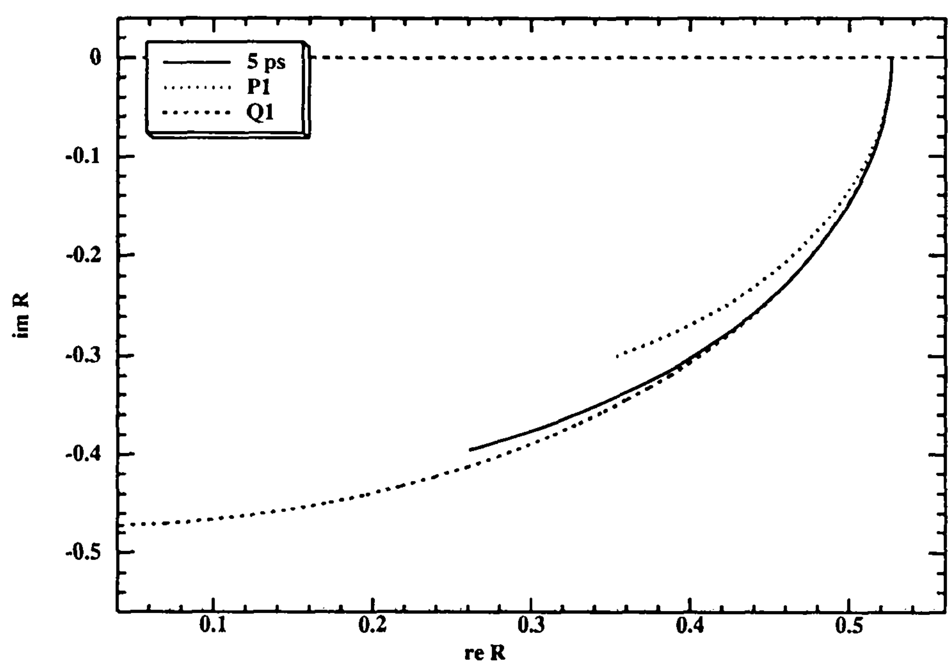

Figure 4.6: Reflection coefficient; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

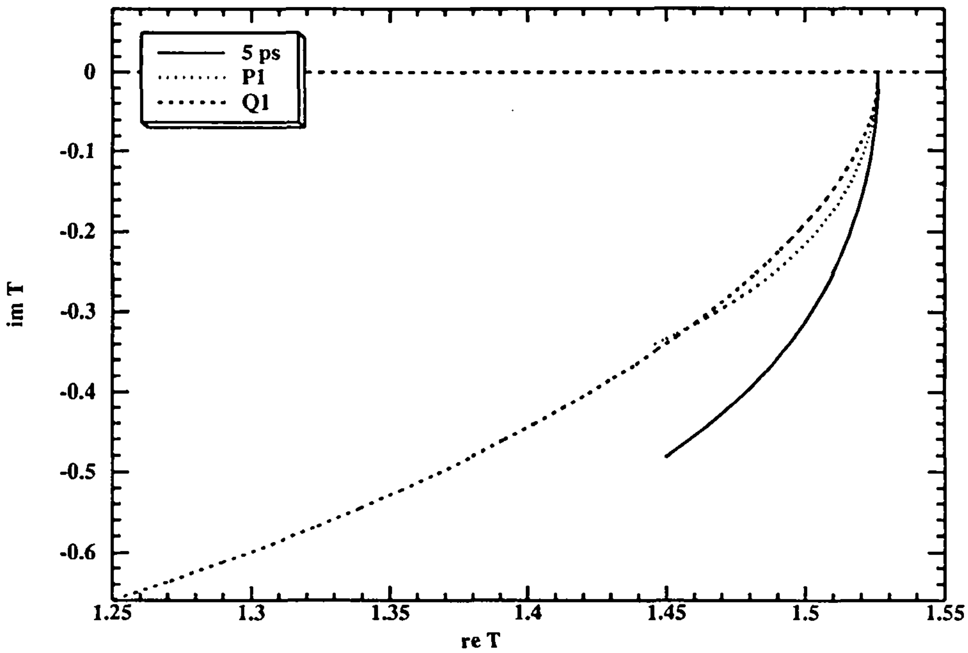

Figure 4.7: Transmission coefficient; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

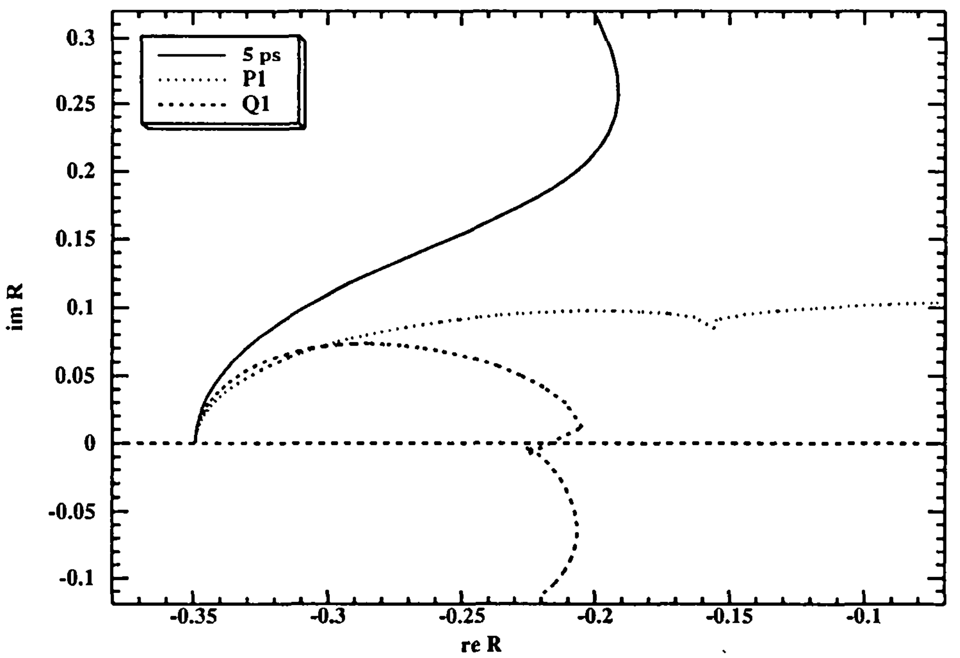

Figure 4.8: Reflection coefficient; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

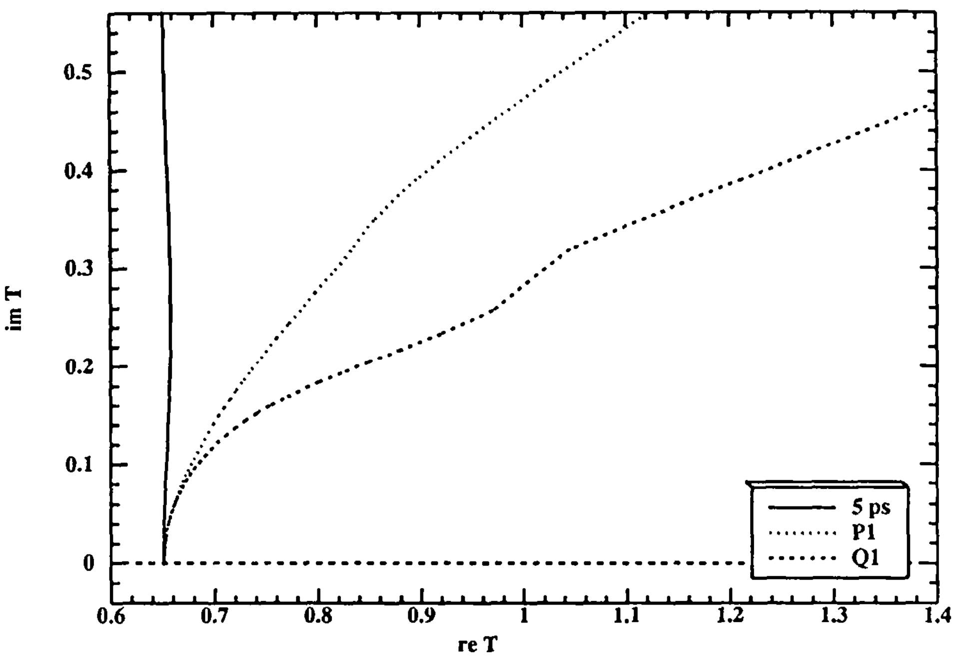

Figure 4.9: Transmission coefficient; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

Of course in the complex plane one does not see clearly the 'parametrization' by h, so let us plot the absolute error of the reflection and transmission coefficients. We chose the absolute error for the following reason. Recall that for the right coefficients we have $|R| \leq 1$ and $1 + R = T$. This means that the relative error of the reflection coefficient is always bigger than the absolute error, but for R close to zero the relative error is not really appropriate. The transmission coefficient T can also have values close to zero, so that the absolute error is also better in that case. Anyway $|T| \leq 2$ and consequently the absolute error is always a reasonable way to represent the error. In figures 4.10, 4.11, 4.12 and 4.13 we show two cases and it is seen that the error remains approximately linear for quite large range of values of h.

Next we consider the difference of the absolute values of exact and approximate coefficients, which is of second order because the first order error terms are purely imaginary (except when there is a total reflection, but then the absolute value of the reflection coefficient is exact anyway). In figures 4.14, 4.15, 4.16 and 4.17 we show these errors and naturally they are considerably smaller than the 'real' errors in the previous pictures.

Then calculating the error terms when also the discrete domains are half planes we first of all have evidently $r_i = t_i$ for all i, and $r_1 = 0$ for all three schemes. We give then the second order error terms for the three cases: first five points scheme, then Q_1 and finally P_1

$$\begin{aligned} r_2 &= \frac{c_1(c_1^2 - c_2^2) z_1 z_2 |k|^2 \cos\alpha (3c_1^2 - 4c_2^2 \sin^2\alpha + 2c_2^2 \sin^4\alpha)}{12 c_2^2 \sqrt{c_1^2 - c_2^2 \sin^2\alpha} (c_1 z_1 \cos\alpha + z_2 \sqrt{c_1^2 - c_2^2 \sin^2\alpha})^2} \\ r_2^q &= \frac{c_1(c_1^2 - c_2^2) z_1 z_2 |k|^2 \cos\alpha (3c_1^2 - 4c_2^2 \sin^2\alpha - 2c_2^2 \sin^4\alpha)}{12 c_2^2 \sqrt{c_1^2 - c_2^2 \sin^2\alpha} (c_1 z_1 \cos\alpha + z_2 \sqrt{c_1^2 - c_2^2 \sin^2\alpha})^2} \\ r_2^p &= \frac{c_1(c_1^2 - c_2^2) z_1 z_2 |k|^2 \cos\alpha (3c_1^2 - 4c_2^2 \sin^2\alpha)}{16 c_2^2 \sqrt{c_1^2 - c_2^2 \sin^2\alpha} (c_1 z_1 \cos\alpha + z_2 \sqrt{c_1^2 - c_2^2 \sin^2\alpha})^2}. \end{aligned}$$

Note that apart the term $\pm 2c_2^2 \sin^4\alpha$ and the factor $3/4 = (\sqrt{3}/2)^2$ the terms are the same and $\sqrt{3}/2$ is the ratio of the distances of the layers of points of different grids in y direction.

Comparing directly the errors of the half plane and 'staircase' cases would not be very interesting because they are of different orders; however, the error in the absolute values are of the same order, so we present some curves showing the errors in both cases for the same scheme. In the half plane case the errors in R and T are so close that they are indistinguishable when plotted so that we show the errors of both R and T in the same picture. In figures 4.18, 4.19, 4.20, 4.21, 4.22 and 4.23 are the results for different cases; *hs* stands for the half space , *sc re* for staircase reflection and *sc tr* for staircase transmission. It is seen that the curves are quite close, they are 'essentially' same. We conclude that the staircase approximation introduces a large phase shift, but 'energetically' it is similar to the half plane case.

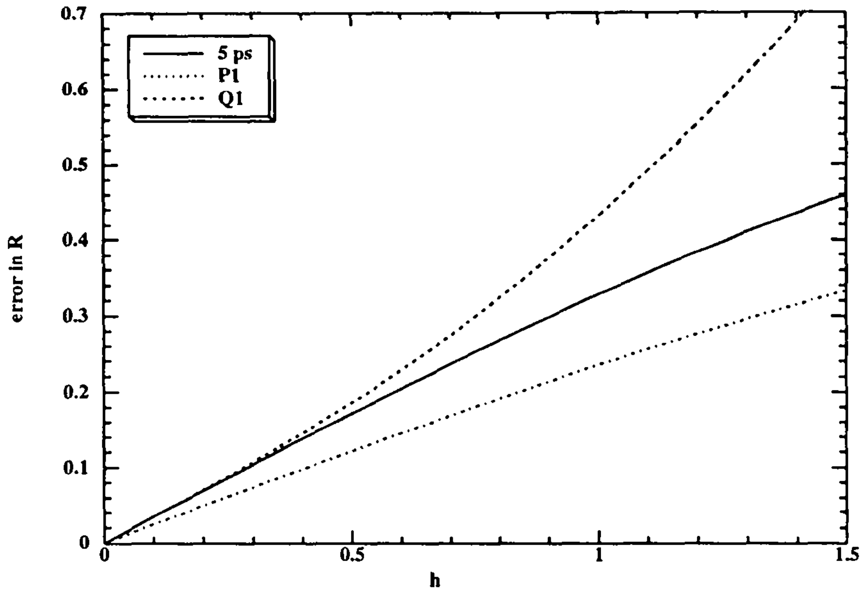

Figure 4.10: Absolute value of the error in reflection coefficient; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

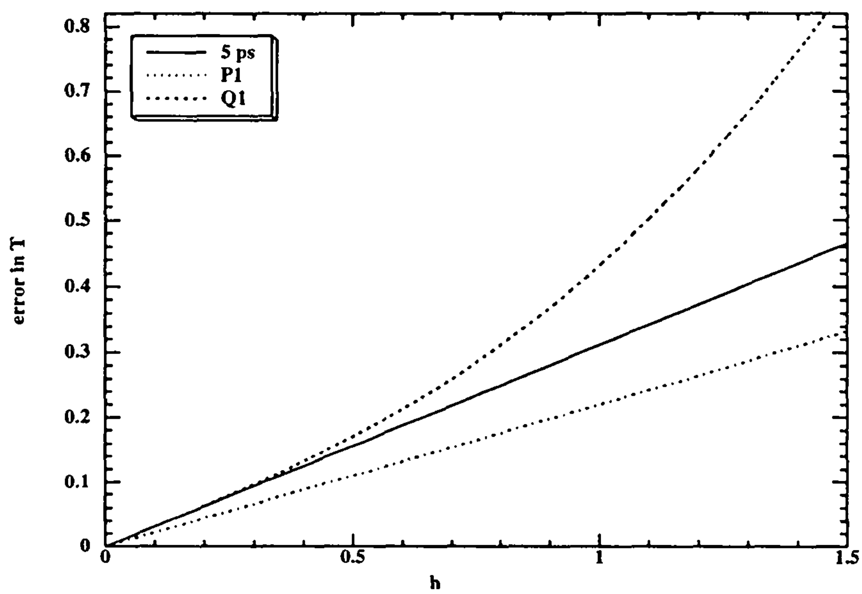

Figure 4.11: Absolute value of the error in transmission coefficient; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

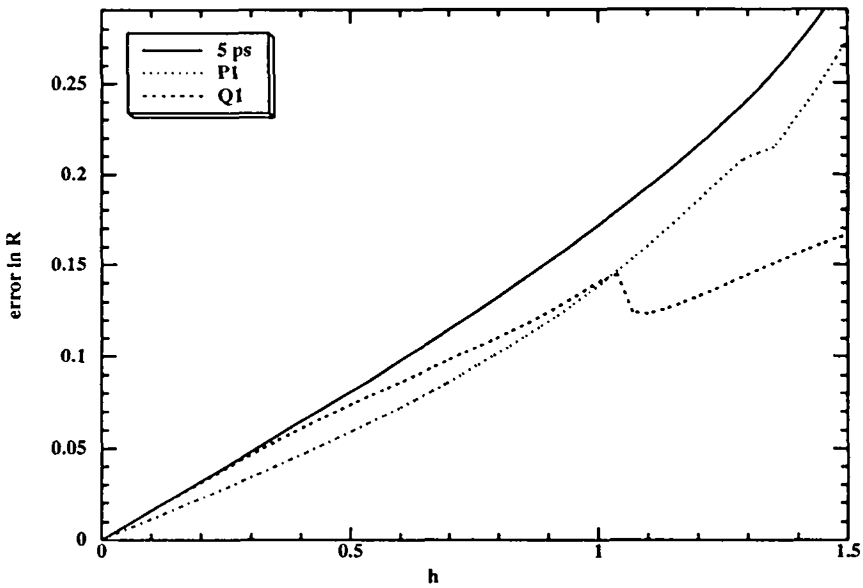

Figure 4.12: Absolute value of the error in reflection coefficient; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

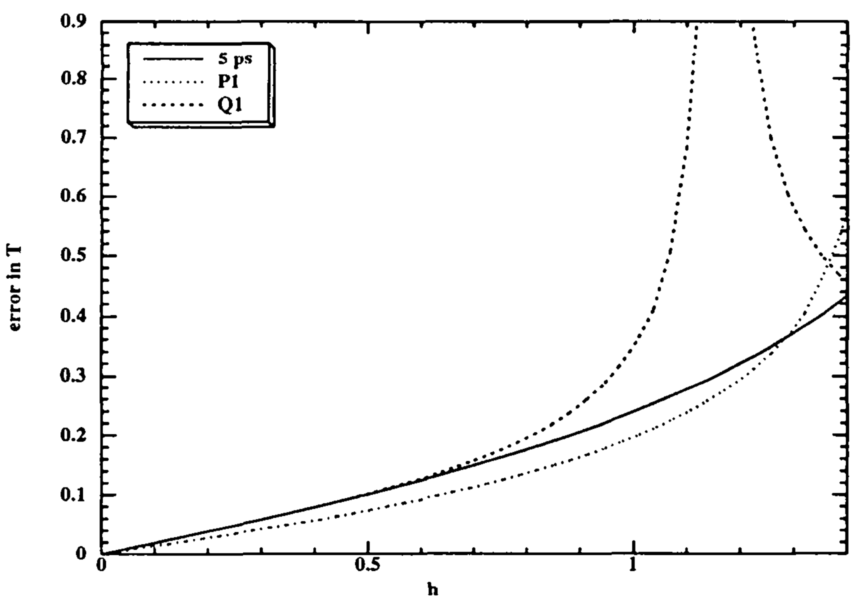

Figure 4.13: Absolute value of the error in transmission coefficient; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

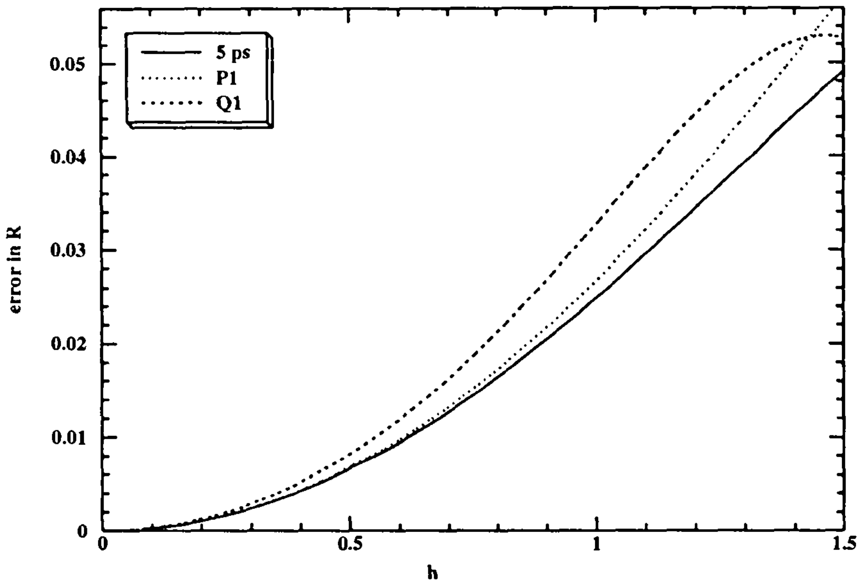

Figure 4.14: Error in absolute value of reflection coefficient; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

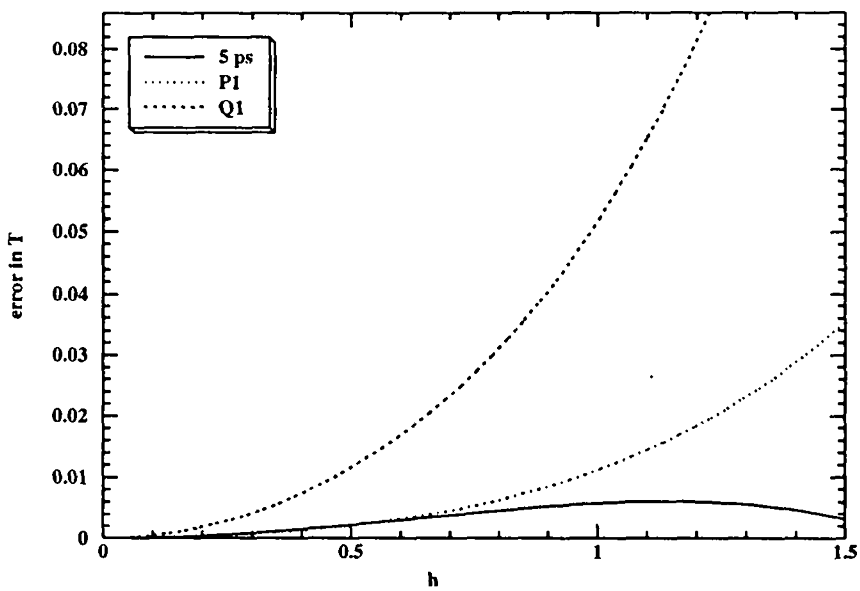

Figure 4.15: Error in absolute value of transmission coefficient; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

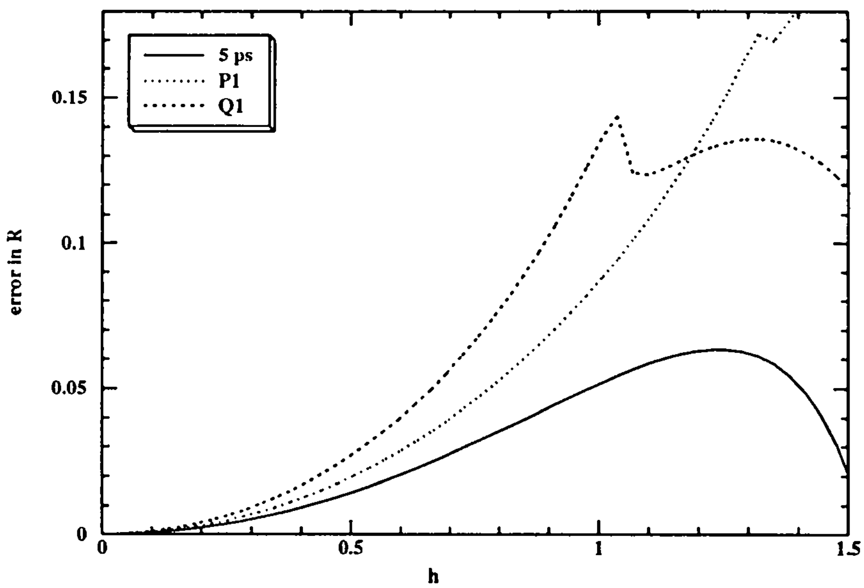

Figure 4.16: Error in absolute value of reflection coefficient; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

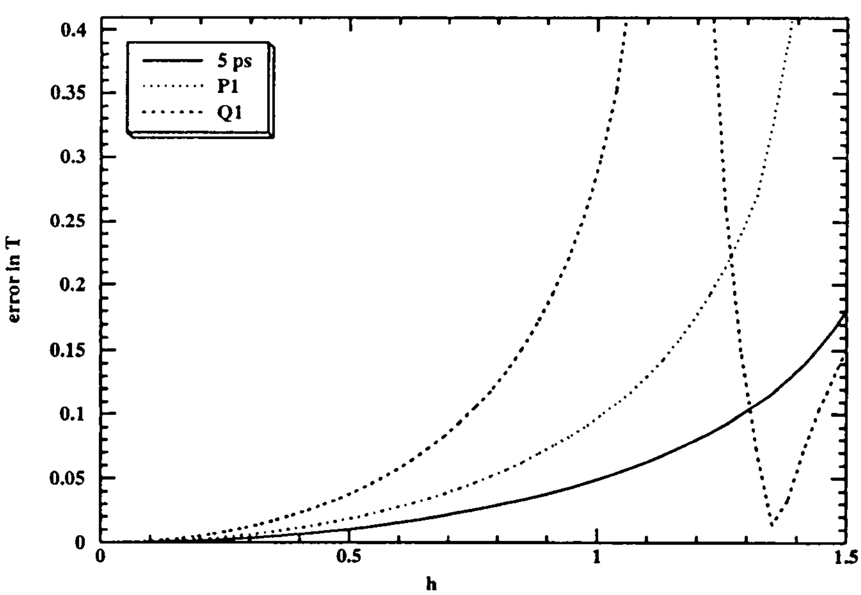

Figure 4.17: Error in absolute value of transmission coefficient; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

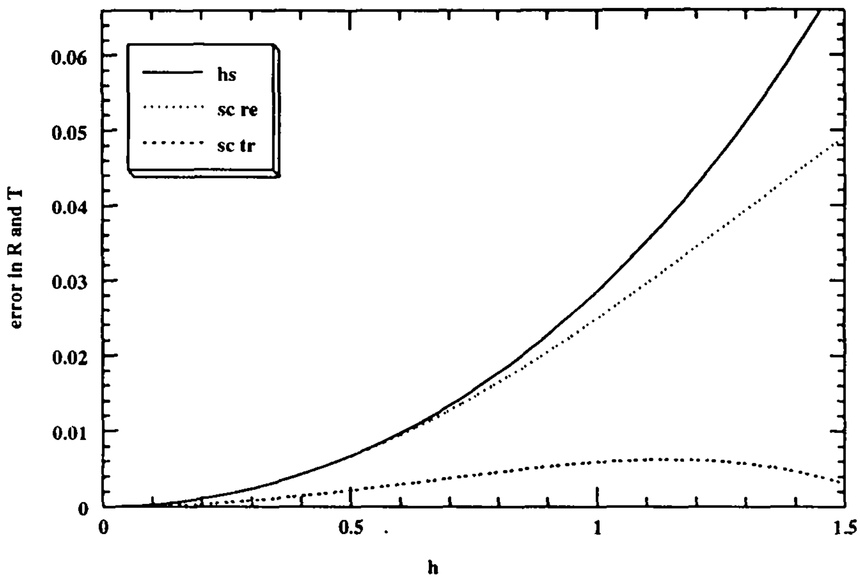

Figure 4.18: Error in absolute value of reflection and transmission coefficient: five points scheme; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

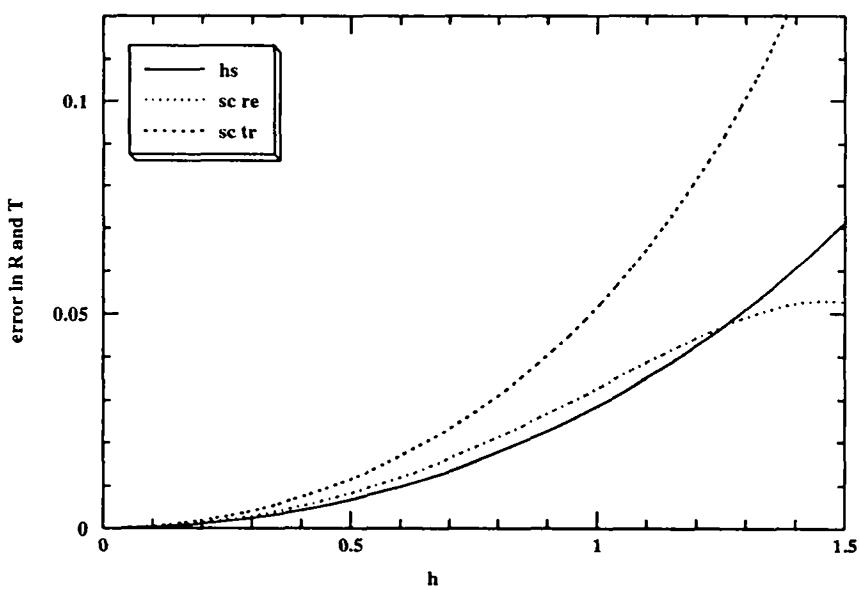

Figure 4.19: Error in absolute value of reflection and transmission coefficient: Q_1 scheme; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

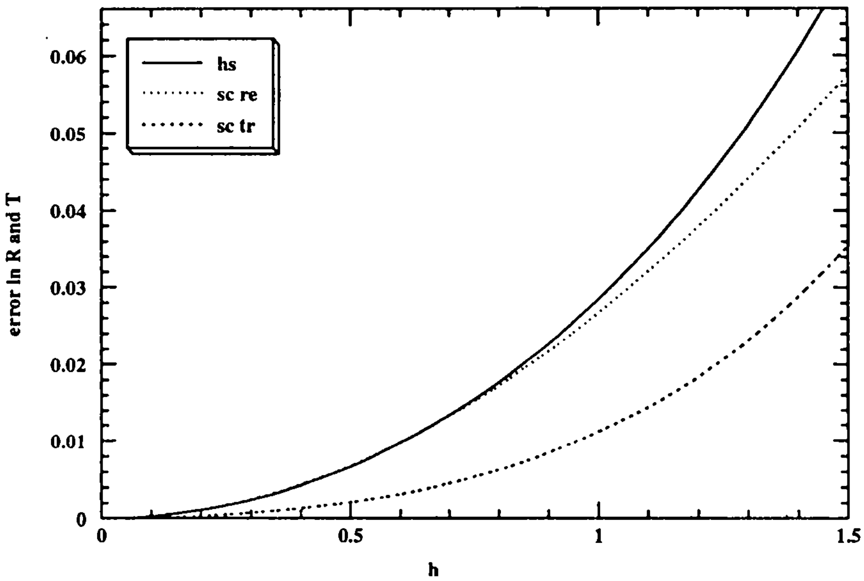

Figure 4.20: Error in absolute value of reflection and transmission coefficient: P_1 scheme; $z_2/z_1 = 0.4$, $c_2/c_1 = 2$ and $\alpha = 20$.

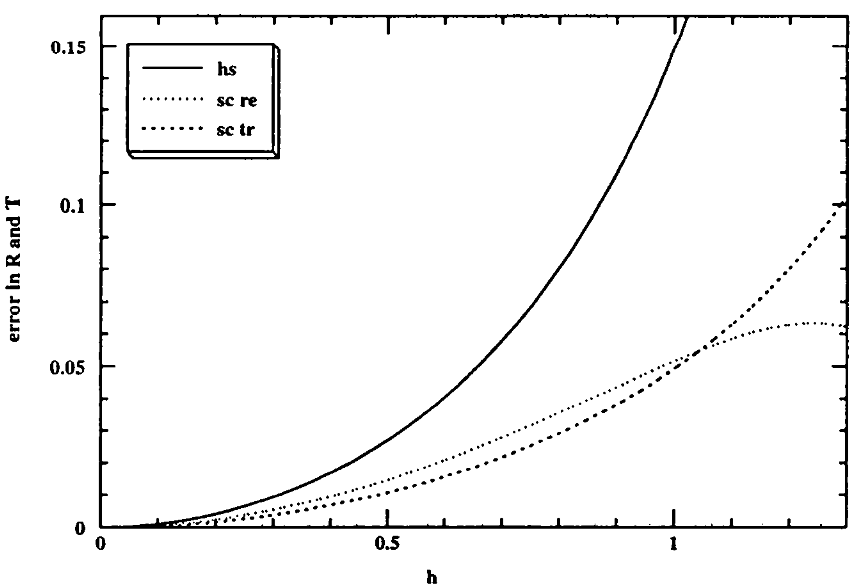

Figure 4.21: Error in absolute value of reflection and transmission coefficient: five points scheme; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

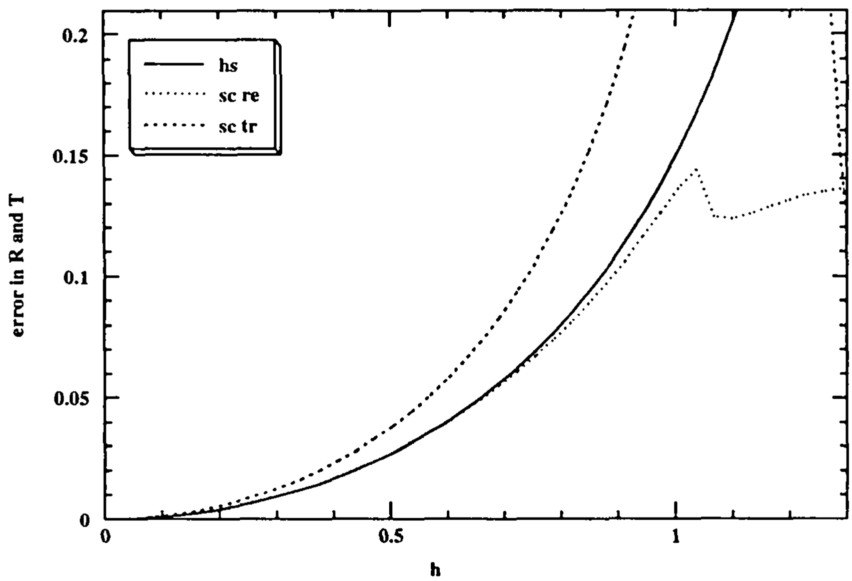

Figure 4.22: Error in absolute value of reflection and transmission coefficient: Q_1 scheme; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

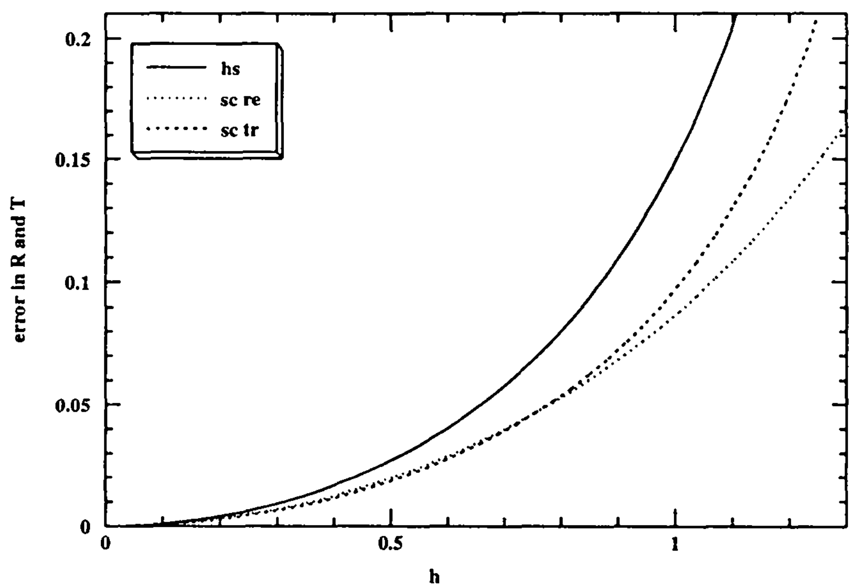

Figure 4.23: Error in absolute value of reflection and transmission coefficient: P_1 scheme; $z_2/z_1 = 1.5$, $c_2/c_1 = 0.6$ and $\alpha = 50$.

5 Conclusion

We have seen that sloppy discretization introduces large errors to the reflection and transmission coefficients. The analysis was made for Q_1, P_1 and five points schemes, but we believe that the results are more generally valid, because many finite difference and finite element schemes would give the equations with the same kind of structure and there appears to be no reason why these other schemes would behave better than the ones we took up. So the overall conclusion seems to be somewhat disappointing: the sloppy discretization is not really acceptable.

One might object that the absolute values are rather accurate; recall that the error in the coefficients was $O(h)$ while the error in the absolute values was $O(h^2)$. In addition the error constant in the absolute value case does not depend very much on the sloppiness. However, in applications where one is mainly interested in the absolute values, it is mostly the absolute value of the *far field* that is desired. Now with finite differences or finite elements it is not possible to calculate the far field (the domain has to be truncated). One solution is to calculate the near field and then use an integral operator (Green function and convolution) to compute the far field. But unfortunately there is no reason why the error in the absolute value of the far field computed this way should be $O(h^2)$.

Acknowledgements I am grateful to Patrick Joly for his interest in this work. Various calculations were done with *Mathematica*, see [WO].

References

[KO] J. A. Kong: *Electromagnetic waves*; Wiley, 1986.

[MI] R. Mittra: *Finite difference time-domain methods*; cours de CEA-EDF-INRIA: *Méthodes asymptotiques et numériques en théorie de la diffusion*; 1990.

[WI] C. Wilcox: *Scattering theory for the d'Alembert equation in exterior domains*; Lecture notes in mathematics 442, Springer, 1975.

[WO] S. Wolfram: *Mathematica: a system for doing mathematics by computer*; Addison-Wesley, 1988.

Appendix A

Here are the listing of the Mathematica session where the calculations of the theorem 4.2 were performed. The intermediate results are not printed because it would easily fill some hundreds of pages. For the same reasons we cannot give all the calculations we have done with Mathematica. To make sense of the listing one should be at least a little familiar with Mathematica (or some other program for symbolic computation), and the easiest way would be to try them on one's own computer.

```
yh1:=1+r1+r2-t1-t2
yh2:=1/a2+a2 r1-r2 a3-t1/a4+t2/a5
yh3:=(8 mu2+4 mu1)(t1+t2)-
     2 mu1(1/a2^2+r1 a2^2+r2 a3^2)-
     2 a1(mu1+mu2)(t1/a4-t2/a5)-
     4 mu2 a1(a4 t1-a5 t2)-2 mu2(a4^2 t1+a5^2 t2)-
     (2 ro2+ro1)(t1+t2)om^2 h^2/.{mu1->c1 z1,
        mu2->c2 z2,ro1->z1/c1,ro2->z2/c2}
yh4:=(8 mu1+4 mu2)(t1/a4-t2/a5)-
     2 mu1(1/a2^3+r1 a2^3-r2 a3^3)-
     2 a1(mu1+mu2)(t1+t2)-
     4 mu1 a1(1/a2^2+r1 a2^2+r2 a3^2)-2 mu2(a4 t1-a5 t2)-
     (2 ro1+ro2)(t1/a4-t2/a5)om^2 h^2/.{mu1->c1 z1,
        mu2->c2 z2,ro1->z1/c1,ro2->z2/c2};

oms=c1^2 k^2+O[h]^2;
a1s=1-3 k^2 si^2 h^2/8+O[h]^4;
a2s=1+I k2 h/2-k2^2 h^2/8+O[h]^3/.k2->k co;
a3s=2-Sqrt[3]+(2 Sqrt[3]-3)(1+2 si^2)k^2 h^2/24+O[h]^3;
a4s=1+I h k sq/(2 c2)-sq^2 k^2 h^2/(8 c2^2)+O[h]^3;
a5s=2-Sqrt[3]+(2 Sqrt[3]-3)(c1^2+2 c2^2 si^2)k^2 h^2/(24 c2^2)+
     O[h]^3;

s1=D[yh3,r1]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s}
s2=D[yh3,r2]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s}
s3=D[yh3,t1]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s,om^2->oms}
s4=D[yh3,t2]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s,om^2->oms}
s5=D[yh4,r1]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s}
s6=D[yh4,r2]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s}
s7=D[yh4,t1]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s,om^2->oms}
s8=D[yh4,t2]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s,om^2->oms}
s9=D[yh2,r1]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s}
s10=D[yh2,r2]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s}
s11=D[yh2,t1]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s}
s12=D[yh2,t2]/.{a1->a1s,a2->a2s,a3->a3s,a4->a4s,a5->a5s};

v1=-1;
v2=-yh2/.{r1->0,r2->0,t1->0,t2->0,a1->a1s,
     a2->a2s,a3->a3s,a4->a4s,a5->a5s};
v3=-yh3/.{r1->0,r2->0,t1->0,t2->0,a1->a1s,
     a2->a2s,a3->a3s,a4->a4s,a5->a5s};
v4=-yh4/.{r1->0,r2->0,t1->0,t2->0,a1->a1s,
     a2->a2s,a3->a3s,a4->a4s,a5->a5s};

Simplify[{v1,v2,v3,v4}];

b=%;

b/.h->0

b0=%;
```

```
Expand[(b-b0)/(I h)/.h->0]

b1=%;

Together[(b-b0-I b1 h)/h^2/.h->0]

b2=%;

ma={{1,1,-1,-1},{s9,s10,s11,s12},
    {s1,s2,s3,s4},{s5,s6,s7,s8}};

Simplify[ma/.h->0]

ma0=%;

Together[NullSpace[Transpose[ma0]]]

vr=%[[1]];

NullSpace[ma0];

vn=%[[1]]

Together[Expand[(ma-ma0)/(I h)/.h->0]]

ma1=%;

ma1[[3,4]]=0;
ma1[[4,4]]=0;

(ma-ma0-I ma1 h)/h^2/.h->0;

Together[%];

Factor[%/.{co^2->1-si^2, 3^(3/2)->3 Sqrt[3],sq^2->c1^2-c2^2 si^2}

ma2=%;

(ma-ma0-I ma1 h)/h^2;

%[[3,4]];

Together[%[[3,3]]];

Together[%/. 3^(3/2)->3 Sqrt[3]]

%27[[4,4]];

Together[%[[3,3]]];

Factor[%/. 3^(3/2)->3 Sqrt[3]]

ma2[[3,4]]=%30;
ma2[[4,4]]=%33;

LinearSolve[ma0,b0]

y0=%;

Together[(b1-ma1.y0).vr/(vr.ma1.vn)]
```

```
la0=%;
Together[y0+la0 vn]
x0=%;
Together[b1-ma1.x0]
d1=%;
LinearSolve[ma0,d1];
Factor[%43[[1]]]
Factor[%43[[2]]]
Factor[%43[[3]]]
Factor[%43[[4]]]
y1={%44,%45,0,%47};
Together[-(b2+ma1.y1-ma2.x0).vr/(vr.ma1.vn)];
Factor[%/.{co^3->co(1-si^2),co^2->1-si^2}]
la1=%;
y1+la1 vn;
Together[%52[[1]]];
Factor[%/.{sq^2->c1^2-c2^2 si^2,co^2->1-si^2}]
Together[%52[[3]]];
Factor[%/.{sq^2->c1^2-c2^2 si^2,co^2->1-si^2}]
```


Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique.

ISSN 0249-6399